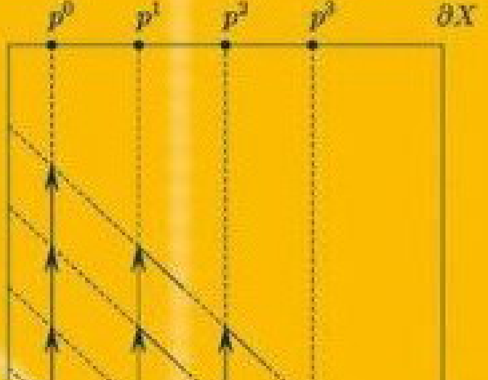
Lecture Notes in Mathematics

1941

Shai M. J. Haran

Arithmetical Investigations

Representation Theory, Orthogonal Polynomials, and Quantum Interpolations



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To Yedidya, Antonia, Elisha, Yehonadav,
Amiad & Yoad.

Preface

This book grew out of lectures given at Kyushu University under the support of the Twenty-first Century COE Program "Development of Dynamical Mathematics with High Functionality" (Program Leader: Prof. Mitsuhiro Nakao). They were meant to serve as a primer to my book [Har5]. Indeed that book is very condense, and hard to read. We included however many new themes, such as the higher rank generalization of [Har5], and the fundamental semigroup. Since the audience consisted mainly of representation theorists, the focus shifted more into representation theory (hence less into geometry). We kept the lecture flair, sometimes explaining basic material in more detail, and sometimes only giving brief descriptions.

This book would have never come to life without the many efforts of Professor Masato Wakayama. The author thanks him also for his incredible hospitality. Thanks are also due to Yoshinori Yamasaki, who did an excellent job of writing down and typing the lectures into LaTeX.

July 2006 Haifa

Contents

0

Introduction: Motivations from Geometry

Summary. In chap. 0 we begin with geometrical motivations and introduction. We recall the analogies between geometry (curve X over a finite field $\mathbb{F}_q$) and arithmetic (number field K), and the two basic problems of arithmetic: the problem of the real primes and the problem of non-existence of a surface $\mathrm{Spec}\mathcal{O}_K \times \mathrm{Spec}\mathcal{O}_K$ (analogues to $X \times_{\mathbb{F}_q} X$). We then give the "Weil philosophy": the explicit sums of arithmetic are the intersection number of Frobenius divisors on the (non-existing, but see [Har6]) surface. This was never made explicit by Weil (and only was spelled out in [Har2]). The proof of the functional equation and the Riemann–Roch in arithmetic give the "Tate philosophy": we are studying the action of the idele-class $\mathbb{A}_K^*/K^*$ on the problematic space $\mathbb{A}/K^*$. The important part of the ergodic action of K^* on the Adele $\mathbb{A}_K$ is encoded in the action of K^* on $\mathbb{A}_K/K$. We then recall the author formula that connects these two philosophies ([Har2], [Har1]), giving the explicit sums in terms of the Fourier transform of the degree $\log |x|_{\mathfrak{p}}^{-1}$.

0.1 Introduction

The main subject of this course is arithmetic. There are many different reasons for which people are attracted to arithmetic. Simple formulation of complicated problems is one of them. Such problems are the Fermat last theorem, the twin primes problem, Goldbach's conjecture and so on. These are very easy to state, however, they are very hard to solve.

There are many similar points between arithmetic and geometry. André Weil says in [We7] that the situation between arithmetic and geometry is like the "Rosetta Stone". It is a big ancient-Egyptian stone in which the same thing was written in three different languages; hieroglyphic, demotic and Greek. Hieroglyphic was used by ancient Egyptians and it had not been known yet. Demotic was used by Arabs including modern Egyptians. Greek was used by Greeks, and other eastern Europeans. Since the last two languages had been well known, we also understood the mysterious first language, Hieroglyphic. As for arithmetic and geometry, correspondings to the above languages are the number fields and the function fields over a finite field $\mathbb{F}_q$, and over the complex numbers $\mathbb{C}$, (that is, compact Riemann surfaces) which are the one-dimensional objects of Geometry. We have tried to understand number fields from the analogies between arithmetic and geometry. Our position is a bit different from the Weil's point of view. Note that on the Rosetta stone there were three different language talking about the same things, but in our case there is one language talking about three different things. We believe that the language we are using is wrong, and there is a "new language" that will unite Arithmetic and Geometry.

The Rosetta Stone	Global Field
Hieroglyphic	Number field
Demotic Greek	Function field $/\mathbb{F}_q$ Function field $/\mathbb{C}$

0.2 Analogies Between Arithmetic and Geometry

Let us begin by reviewing the analogy between arithmetic and geometry. In arithmetic, we start from the ring of integers $\mathbb{Z}$. The ring $\mathbb{Z}$ is included in its fraction field, the field of the rational numbers $\mathbb{Q} = \mathrm{Frac}(\mathbb{Z})$. In geometric, the basic object is the ring of polynomials $k[x]$ in one variable over a field k included in the field of rational functions $k(x) = \mathrm{Frac}(k[x])$. These rings $\mathbb{Z}$ and $k[x]$ have many common properties. For example, they have the division with remainder principal, are PID and are UFD, that is, every element can be uniquely written as a product of irreducible elements. Take an irreducible polynomial $f \in k[x]$. Then $k[x]$ embeds in the local ring $k_f[[f]] := \varprojlim k[x]/(f^n)$, the ring of formal power series in f. Here $k_f := k[x]/(f)$ is the residue field. Namely every element in $k[x]$ can be written as a power series in f. Also $k_f[[f]] \subset k_f((f))$ where $k_f((f))$ is the field of formal Laurent series in f. For example if we take $k = \mathbb{C}$ (or any algebraically closed field), an irreducible polynomial can be written as $f(x) = x - \alpha$ for some $\alpha \in \mathbb{C}$. Since $\mathbb{C}_f = \mathbb{C}$, every rational function can be expressed as a Laurent series in $(x - \alpha)$. Let p be a prime. In arithmetic side, the ring of p-adic integers corresponds to $k_f[[f]]$. Every rational integer is represented as a power series in p and we obtain $\mathbb{Z}_p := \varprojlim \mathbb{Z}/(p^n)$. Similarly $\mathbb{Q} \subset \mathbb{Q}_p$, the field of p-adic numbers.

In geometry we have two types of geometry; affine and projective geometry. If $k = \mathbb{C}$, every rational function is written as a Laurent series of $\frac{1}{x}$, that is, $k(x) \subset k((\frac{1}{x}))$. Projectively speaking, a rational function can be expanded at the "infinite point" ∞. Then ∞ clearly corresponds to the ring of formal power series $k[[\frac{1}{x}]]$ in $\frac{1}{x}$. In arithmetic this resembles to the inclusion of $\mathbb{Q}$ into the completion $\mathbb{R} = \mathbb{Q}_\eta$ of $\mathbb{Q}$ at the "real prime η". Then a problem occurs; what is $\mathbb{Z}_\eta$? For a finite prime $p \neq \eta$, the p-adic integers $\mathbb{Z}_p$ is given by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \,|\, |x|_p \leq 1\}$ where $|\cdot|_p$ is the p-adic absolute value. From this point of view, $\mathbb{Z}_\eta$ is considered as the interval $[-1, 1]$, however, it is not closed under addition and is not given by any inverse limit neither.

Function fields (Geometry)	Number fields (Arithmetic)
$k[x] \subset k(x)$	$\mathbb{Z} \subset \mathbb{Q}$
$k_f[[f]] \subset k_f((f)) \qquad k[[\frac{1}{x}]] \subset k((\frac{1}{x}))$	$\mathbb{Z}_p \subset \mathbb{Q}_p \qquad$ "$\mathbb{Z}_\eta$" $\subset \mathbb{Q}_\eta = \mathbb{R}$

We further remark that there is an obvious difference between number fields and function fields from the point of the tensor product. In geometry one can take a product of given geometrical object and obtain a new geometrical object. For example, the product of the affine line $\mathbb{A}^1$ with itself is the plane. This correspond to the tensor product of two polynomial rings $k[x_1]$ and $k[x_2]$. The product $k[x_1] \otimes_k k[x_2]$ in the category of k-algebra is equal to $k[x_1, x_2]$, ring of polynomials in two variables. On the other hand, if we consider the tensor product of two copies of $\mathbb{Z}$, taken in the category of commutative rings we obtain only $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$.

0.3 Zeta Function for Curves

Let X be a (smooth, projective) curve of genus g defined over the finite field $k = \mathbb{F}_q$. Let $K = k(X)$ be the field of k-rational functions and $\mathfrak{p}$ be a maximal ideal of the coordinate ring $k[X]$. We denote by $K_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ the completion of K with respect to $\mathfrak{p}$ and the ring of integers of $K_\mathfrak{p}$, respectively. Let $K_\mathfrak{p}^* = K_\mathfrak{p} \backslash \{0\}$ and $\mathcal{O}_\mathfrak{p}^*$ be the unit group of $\mathcal{O}_\mathfrak{p}$. Let $\phi_\mathfrak{p}$ be the characteristic function of $\mathcal{O}_\mathfrak{p}$ and $dx_\mathfrak{p}$ (resp. $d^*x_\mathfrak{p}$) be the additive (resp. multiplicative) Haar measure on $K_\mathfrak{p}$ (resp. $K_\mathfrak{p}^*$) normalized by $dx_\mathfrak{p}(\mathcal{O}_\mathfrak{p}) = 1$ (resp. $d^*x_\mathfrak{p}(\mathcal{O}_\mathfrak{p}^*) = 1$). We denote by $\mathbb{A}$ (resp. $\mathbb{A}^*$) the adele ring (resp. idele group) of K. Put $\mathcal{O}_\mathbb{A} := \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}$, $\mathcal{O}_\mathbb{A}^* := \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^*$, $dx := \bigotimes_\mathfrak{p} dx_\mathfrak{p}$ and $d^*x := \bigotimes_\mathfrak{p} d^*x_\mathfrak{p}$. Then the zeta function for X is defined by

$$\text{(i)} \qquad \zeta_X(s) := \exp\Big(\sum_{n \geq 1} \#X(k_n) \frac{q^{-sn}}{n}\Big),$$

where $k_n := \mathbb{F}_{q^n}$. Let $k(\mathfrak{p}) := \mathcal{O}_\mathfrak{p}/\mathfrak{p}$ be the residue field and $\mathbb{N}\mathfrak{p} := \#k(\mathfrak{p}) = q^{\deg \mathfrak{p}}$ with $\deg \mathfrak{p} := [k(\mathfrak{p}) : k]$. Then we have the following calculations:

$$\text{(ii)} \qquad \zeta_X(s) = \prod_\mathfrak{p} \big(1 - \mathbb{N}\mathfrak{p}^{-s}\big)^{-1}$$

$$\text{(iii)} \qquad = \sum_{\mathfrak{a} \geq 0} \mathbb{N}\mathfrak{a}^{-s}$$

$$= \int_{\mathbb{A}^*/K^*\mathcal{O}_\mathbb{A}^*} \Big(\sum_{\gamma \in K^*/k^*} \phi_\mathbb{A}(\gamma\mathfrak{a})\Big) |\mathfrak{a}|_\mathbb{A}^s d^*\mathfrak{a} = \int_{\mathbb{A}^*/\mathcal{O}_\mathbb{A}^*} \phi_\mathbb{A}(\mathfrak{a}) |\mathfrak{a}|_\mathbb{A}^s d^*\mathfrak{a}$$

$$\text{(iv)} \qquad = \prod_\mathfrak{p} \int_{K_\mathfrak{p}^*/\mathcal{O}_\mathfrak{p}^*} \phi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}) |\mathfrak{a}_\mathfrak{p}|_\mathfrak{p}^s d^*\mathfrak{a}_\mathfrak{p}$$

$$\text{(v)} \qquad = \sum_{\mathfrak{a}\in\mathrm{Pic}_K} \frac{q^{h^0(\mathfrak{a})}-1}{q-1} q^{-s\cdot\deg(\mathfrak{a})}$$

$$\text{(vi)} \qquad = \frac{\prod_{i=1}^{2g}(1-\lambda_i q^{-s})}{(1-q^{-s})(1-q^{1-s})}.$$

Here $\phi_{\mathbb{A}}$ is the characteristic function of $\mathcal{O}_{\mathbb{A}}$, $|\mathfrak{a}|_{\mathbb{A}} := \prod_{\mathfrak{p}} |\mathfrak{a}_{\mathfrak{p}}|_{\mathfrak{p}}$ and $|\cdot|_{\mathfrak{p}}$ is the normalized absolute value on $K_{\mathfrak{p}}$ as $d(ax_{\mathfrak{p}}) = |a|_{\mathfrak{p}} \cdot dx_{\mathfrak{p}}$ and $|\pi|^s = \mathbb{N}\mathfrak{p}^{-s}$ with $\mathfrak{p} = (\pi)$. It is easy to check these equalities. In fact, one obtains (0.3) (i) $\iff$ (0.3) (ii) by taking $d\log$ and the fact

$$\#X(k_n) = \sum_{\substack{\mathfrak{p} \\ \deg\mathfrak{p}|n}} \deg\mathfrak{p} = \sum_{k(\mathfrak{p})\subseteq k_n} \deg\mathfrak{p}$$

since each $\mathfrak{p}$ with $k(\mathfrak{p}) \subseteq k_n$ gives $\deg\mathfrak{p}$ points in $X(k_n)$. By the unique factorization, we have (0.3) (ii) $\iff$ (0.3) (iii). To show the equalities (0.3) (iii) $\iff$ (0.3) (iv) $\iff$ (0.3) (v), recall that

$$\mathrm{Div}_K = \mathbb{A}^*/\mathcal{O}_{\mathbb{A}}^*, \qquad \mathrm{Pic}_K = \mathbb{A}^*/\mathcal{O}_{\mathbb{A}}^* K^* \xrightarrow{|\cdot|_{\mathbb{A}}} q^{\mathbb{Z}}$$

and the kernel $\mathbb{A}^{(1)}/\mathcal{O}_{\mathbb{A}}^* K^* = \mathrm{Pic}_K^{(1)}$ is finite. Let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$ be the representative of $\mathrm{Pic}_K^{(1)}$. Let $\mathfrak{c} \in \mathrm{Pic}_K$ of degree 1. Then for any $\mathfrak{a} \in \mathrm{Div}_K$, we can write $\mathfrak{a} = f \cdot \mathfrak{a}_i \cdot \mathfrak{c}^n$ with some $f \in K^*/k^*$ and $n = \deg\mathfrak{a}$. Then we have

$$\mathfrak{a} \geq 0 \iff (f) \geq -\mathfrak{a}_i - n\mathfrak{c} \iff f \in H^0(X, \mathcal{O}_{\mathbb{A}}(\mathfrak{a}_i + n\mathfrak{c})) \iff \phi_{\mathbb{A}}(f\cdot\mathfrak{a}_i\cdot\mathfrak{c}^n) = 1.$$

Hence the number of such $\mathfrak{a}$ is equal to

$$\frac{q^{h^0(\mathfrak{a}_i+n\mathfrak{c})}-1}{q-1} = \sum_{f\in K^*/k^*} \phi_{\mathbb{A}}(f\cdot\mathfrak{a}_i\cdot\mathfrak{c}^n),$$

where

$$h^0(\mathfrak{a}) := \dim H^0(X, \mathcal{O}_{\mathbb{A}}(\mathfrak{a})) = \frac{1}{\log q} \log \sum_{\gamma\in K} \phi_{\mathbb{A}}(\gamma\mathfrak{a}).$$

Therefore, using the formula $|\mathfrak{a}|_{\mathbb{A}} = q^{-\deg\mathfrak{a}}$ (or $\frac{\log|\mathfrak{a}|_{\mathbb{A}}^{-1}}{\log q} = \deg\mathfrak{a}$), we obtain the desired equalities. The shape (0.3) (vi) of $\zeta_X(s)$ follows from the Riemann–Roch theorem. Comparing the formula (0.3) (i) and (0.3) (vi) and taking $d\log$, we have

$$\#X(k_n) = \underbrace{1+q^n}_{\#\mathbb{P}^1(k_n)} - \sum_{i=1}^{2g} \lambda_i^n \qquad (n \geq 1).$$

0.4 The Riemann–Roch Theorem

Let $\psi_{\mathfrak{p}}$ be a character of $K_{\mathfrak{p}}$ which is trivial on $\mathcal{O}_{\mathfrak{p}}$ but non-trivial on $\mathfrak{p}^{-1}\mathcal{O}_{\mathfrak{p}}$, and define

$$\mathcal{F}_{\mathfrak{p}}\varphi(y) := \int_{K_{\mathfrak{p}}} \varphi(x)\psi_{\mathfrak{p}}(xy)dx_{\mathfrak{p}}.$$

Notice that $\psi_{\mathfrak{p}}$ is unique mod $\mathcal{O}_{\mathfrak{p}}^*$, and the above normalization is equivalent to $\mathcal{F}_{\mathfrak{p}}\phi_{\mathfrak{p}} = \phi_{\mathfrak{p}}$. Similarly we denote by $\psi := \bigotimes_{\mathfrak{p}} \psi_{\mathfrak{p}}$ the character of $\mathbb{A}/\mathcal{O}_{\mathbb{A}}$ and set

$$\mathcal{F}\varphi(y) := \int_{\mathbb{A}} \varphi(x)\psi(xy)dx.$$

Let $\partial = \partial_\psi \in \mathbb{A}^*$ such that $\psi(\partial K) = 1$. Then we have $(\mathbb{A}/K)^\wedge \simeq K$ via $\psi(\partial\gamma x) \longrightarrow \gamma$. The Riemann–Roch theorem for X asserts that

$$q^{g-1}\sum_{\gamma\in K}\phi_{\mathbb{A}}(\gamma\mathfrak{a}) = |\mathfrak{a}|_{\mathbb{A}}^{-1}\sum_{\gamma\in K}\phi_{\mathbb{A}}(\gamma\partial\mathfrak{a}^{-1}). \tag{0.1}$$

Taking $\frac{1}{\log q}\log(\cdot)$, we have

$$h^0(\mathfrak{a}) = \deg\mathfrak{a} + h^0(\partial - \mathfrak{a}) + 1 - g. \tag{0.2}$$

We give two proofs.

Proof. I: Consider the function

$$\Phi_{\mathfrak{a}}(x) := \sum_{\gamma\in K}\phi_{\mathbb{A}}(\mathfrak{a}(x+\gamma)).$$

We view $\Phi_{\mathfrak{a}}$ as an operator on $L_2(\mathbb{A}/K, dx)$ via convolution. We calculate its trace in two different ways

$$\text{geometric}: \ \mathrm{Tr}\Phi_{\mathfrak{a}} = \int_{\mathbb{A}/K}\Phi_{\mathfrak{a}}(0)dx = dx(\mathbb{A}/K)\sum_{\gamma\in K}\phi_{\mathbb{A}}(\gamma\mathfrak{a}),$$

and $dx(\mathbb{A}/K) = q^{g-1}$. On the other hand, we have an orthogonal basis $\Psi_\gamma(x) = \frac{\psi(\partial\gamma x)}{||\psi(\partial\gamma x)||}$ $(\gamma \in K)$ for $L_2(\mathbb{A}/K, dx)$. Since $\Phi_{\mathfrak{a}}\psi_\gamma = \mathcal{F}(\phi_{\mathbb{A}}(\mathfrak{a}x))(-\partial\gamma)\cdot\psi_\gamma = |\mathfrak{a}|_{\mathbb{A}}^{-1}\cdot\phi_{\mathbb{A}}(\partial\mathfrak{a}^{-1}\gamma)\cdot\psi_\gamma$, we have

$$\text{spectral}: \ \mathrm{Tr}\Phi_{\mathfrak{a}} = |\mathfrak{a}|_{\mathbb{A}}^{-1}\sum_{\gamma\in K}\phi_{\mathbb{A}}(\partial\mathfrak{a}^{-1}\gamma).$$

Therefore we obtain the desired formula (0.1). □

Proof. II: Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{A}^*/\mathcal{O}_{\mathbb{A}}^*$ with $\mathfrak{a} \cdot \mathcal{O}_{\mathbb{A}} \subseteq \mathfrak{b} \cdot \mathcal{O}_{\mathbb{A}}$. Then we have the following exact sequence

$$0 \longrightarrow \mathfrak{a} \cdot \mathcal{O}_{\mathbb{A}} \cap K \longrightarrow \mathfrak{b} \cdot \mathcal{O}_{\mathbb{A}} \cap K \longrightarrow \mathfrak{b} \cdot \mathcal{O}_{\mathbb{A}}/\mathfrak{a} \cdot \mathcal{O}_{\mathbb{A}} \longrightarrow \mathbb{A}/(\mathfrak{a} \cdot \mathcal{O}_{\mathbb{A}} + K) \longrightarrow \mathbb{A}/(\mathfrak{b} \cdot \mathcal{O}_{\mathbb{A}} + K) \longrightarrow 0.$$

Then replacing $\mathfrak{a}$ (resp. $\mathfrak{b}$) with $\mathfrak{a}^{-1}$ (resp. $\mathfrak{b}^{-1}$) and taking $\dim_{\mathbb{F}_q}(\cdot)$, we have

$$0 = h^0(\mathfrak{a}) - h^0(\mathfrak{b}) + (\deg\mathfrak{b} - \deg\mathfrak{a}) - h^1(\mathfrak{a}) + h^1(\mathfrak{b})$$

or

$$\deg\mathfrak{a} - h^0(\mathfrak{a}) + h^1(\mathfrak{a}) \equiv (\text{const.}) = g - 1.$$

Since we have also the perfect duality:

$$\left[\partial^{-1}\mathfrak{a}\mathcal{O}_{\mathbb{A}} \cap K\right] \times \mathbb{A}/(\mathfrak{a}^{-1}\mathcal{O}_{\mathbb{A}} + K) \longrightarrow \mathbb{C}^{(1)}; \qquad (\gamma, x) \longmapsto \psi(\partial\gamma x),$$

we have

$$h^1(\mathfrak{a}) = \dim\mathbb{A}/(\mathfrak{a}^{-1}\mathcal{O}_{\mathbb{A}} + K) = \dim\partial^{-1}\mathfrak{a}\mathcal{O}_{\mathbb{A}} \cap K = h^0(\partial - \mathfrak{a}).$$

Hence we obtain (0.2). □

As corollaries of the formula (0.2), we have

$$\begin{aligned} \mathfrak{a} = 0 &: h^0(0) = 1,\ h^0(\partial) = g,\ \chi(0) = 1 - g, \\ \mathfrak{a} = \partial &: \deg\partial = 2g - 2 \end{aligned}$$

and

$$\deg\mathfrak{a} > 2g - 2 \Longrightarrow h^0(\partial - \mathfrak{a}) = 0 \Longrightarrow h^0(\mathfrak{a}) = \deg\mathfrak{a} + 1 - g.$$

This gives the rationality of zeta function $\zeta_X(s)$ and the shape (0.3) (vi). Namely, using (0.3) (v), we have

$$\begin{aligned} \zeta_X(s) = &\ (\text{polynomial of degree} \leq 2(g-1)) \\ &+ \frac{q^{(2g-1)(1-s)}}{q-1} \cdot \frac{1}{1-q^{1-s}} - \frac{q^{(2g-1)(-s)}}{q-1} \cdot \frac{1}{1-q^{-s}}. \end{aligned}$$

Further, using the Riemann–Roch theorem (0.1) in (0.3) (iii) (i.e., we divide the integral as $|\mathfrak{a}|_{\mathbb{A}} \leq 1$ and $|\mathfrak{a}|_{\mathbb{A}} > 1$ and use the Riemann–Roch theorem to change $|\mathfrak{a}|_{\mathbb{A}} > 1$ to $|\mathfrak{a}|_{\mathbb{A}} \leq 1$. Care the term $\gamma = 0$), we get the holomorphic continuation of $\zeta_X(s)$ to the whole plane $\mathbb{C}$ except for simple poles at $s = 0, 1$ with

$$\frac{1}{\log q}\operatorname*{Res}_{s=0}\zeta_X(s) = -\frac{h}{q-1}, \qquad \frac{1}{\log q}\operatorname*{Res}_{s=1}\zeta_X(s) = \frac{h}{q-1}q^{1-g} = \frac{h}{q-1}|\partial|_{\mathbb{A}}^{\frac{1}{2}},$$

and also obtain the functional equation

$$\zeta_X(s) = q^{2(g-1)(\frac{1}{2}-s)}\zeta_X(1-s) = |\partial|_{\mathbb{A}}^{s-\frac{1}{2}}\zeta_X(1-s).$$

The Riemann hypothesis for the zeta function $\zeta_X(s)$ can be stated in various ways:

(a) $$\zeta_X(s) = 0 \Longrightarrow \mathrm{Re}(s) = \frac{1}{2},$$

(b) $$|\lambda_i| = q^{\frac{1}{2}},$$

(c) $$|\#X(k_n) - q^n| = O(q^{\frac{n}{2}}) \qquad (n \to \infty),$$

(d) $$\sum_i f(\lambda_i) f(q/\overline{\lambda_i}) \geq 0 \qquad (f \in \mathbb{Z}[T, T^{-1}]),$$

(e) $$\sum_{\substack{\zeta_X(s)=0 \\ \mathrm{mod}\,\frac{2\pi i}{\log q}\mathbb{Z}}} \widehat{f}(s)\overline{\widehat{f}(1-\overline{s})} \geq 0 \qquad (f \in C_c(g^{\mathbb{Z}}),\ \widehat{f}(s) := \sum_n f(g^n) q^{ns}).$$

Notice that these formulas are all equivalent. Let $f^\natural(g^n) := q^{-n}\overline{f}(g^{-n})$ for $f \in C_c(g^{\mathbb{Z}})$. Since

$$(f_1 * f_2)^\wedge(s) = \widehat{f_1}(s) \cdot \widehat{f_2}(s), \qquad (f^\natural)^\wedge(s) = \overline{\widehat{f}(1-\overline{s})},$$

the formula (e) is equivalent to the following

(f) $$W(f * f^\natural) \geq 0,$$

where

$$W(f) := \sum_{\substack{\zeta_X(s)=0 \\ \mathrm{mod}\,\frac{2\pi i}{\log q}\mathbb{Z}}} \widehat{f}(s) = \sum_n f(g^n) \sum_{i=1}^{2g} \lambda_i^n = \sum_n f(g^n)\big(1 + q^n - \#X(k_n)\big).$$

0.5 The Castelnuovo–Severi Inequality

We have a surface $X \times X$ and the Frobenius divisor $\mathfrak{f}^n = \{(x, x^{q^n})\}$ on $X \times X$. We see that $\mathfrak{f}^n$ is inseparable, $d\mathfrak{f}^n = 0$ and $d(\mathrm{id} - \mathfrak{f}^n) = \mathrm{id}$. Let Δ be the diagonal of $X \times X$. The intersection points of $\mathfrak{f}^n \cap \Delta$ are all simple and

$$\langle \mathfrak{f}^n, \Delta \rangle = \#X(k_n) = 1 + q^n - \sum_{i=1}^{2g} \lambda_i^n,$$

$$\langle \mathfrak{f}^n, \Delta \rangle = \sum_{\mathfrak{p}} \langle \mathfrak{f}^n, \Delta \rangle_{\mathfrak{p}} \sim \begin{cases} 0 & \text{if } \deg\mathfrak{p} \nmid n, \\ \deg\mathfrak{p} & \text{if } \deg\mathfrak{p} | n. \end{cases}$$

Note also that, taking $n = 0$, we have "$\#X(\mathbb{F}_1)$"$= \langle \Delta, \Delta \rangle = 2(1-g)$. Similarly, setting $\mathfrak{f}^{-n} = q^{-n}\{(x^{q^n}, x)\}$, we have

$$\langle \mathfrak{f}^{-n}, \Delta \rangle = 1 + q^{-n} - \sum_{i=1}^{2g} \lambda_i^{-n} = q^{-n}\Big(1 + q^n - \sum_{i=1}^{2g} \lambda_i^n\Big).$$

Hence we have

$$W(f) = \widehat{f}(0) + \widehat{f}(1) - \sum_{n\in\mathbb{Z}} f(g^n) \langle \mathfrak{f}^n, \Delta \rangle$$
$$= \langle f(\mathfrak{f}), \delta_0 + \delta_\infty - \Delta \rangle ,$$

where $f(\mathfrak{f}) := \sum_{n\in\mathbb{Z}} f(g^n)\mathfrak{f}^n$ and δ_0 (resp. δ_∞) is $X \times \mathrm{pt}$ (resp. $\mathrm{pt} \times X$). We see that the Riemann hypothesis in the form (f) is equivalent to the fundamental inequality

$$\langle f * f^\natural, \Delta \rangle \leq \langle f * f^\natural, \delta_0 + \delta_\infty \rangle , \quad \text{or, to the}$$

Castelnuovo–Severic inequality : $\quad \langle f, f \rangle \leq 2 \langle f, \delta_0 \rangle \langle f, \delta_\infty \rangle = 2\widehat{f}(0)\widehat{f}(1).$

Note that we can take f $\mathbb{Z}$-valued, so that $f(\mathfrak{f})$ is a divisor on $X \times X$.

More generally, let X and Y be curves of genus g_X and g_Y, respectively. Then $V := X \times Y$ is a surface. Take $f \in \mathrm{Div}V$ and put $\delta_X := \langle f, X \times \mathrm{pt} \rangle$ and $\delta_Y := \langle f, \mathrm{pt} \times Y \rangle$. Then we have the Castelnuovo–Severi inequality:

$$\frac{1}{2} \langle f, f \rangle \leq \delta_X \cdot \delta_Y. \tag{0.3}$$

Let us prove this inequality (following [MT]). By the Riemann–Roch theorem for V, we have

$$\chi(f) = h^0(f) - h^1(f) + h^0(\partial_V - f) = \frac{1}{2} \langle f, f - \delta_V \rangle + \chi(\mathcal{O}_V),$$

where $\partial_V = \partial_X \times Y + X \times \partial_Y$. Note that $\chi(\mathcal{O}_V) = \chi(\mathcal{O}_X)\chi(\mathcal{O}_Y) = (1 - g_X)$ $(1 - g_Y)$. We have

$$\langle f, \partial_V \rangle = (\mathrm{deg}\partial_X)\delta_Y + \delta_X(\mathrm{deg}\partial_Y) = 2(g_X - 1) \cdot \delta_Y + \delta_X \cdot 2(g_Y - 1).$$

Therefore we obtain

$$\chi(f) = \frac{1}{2} \langle f, f \rangle + (1 - g_X)\delta_Y + \delta_X(1 - g_Y) + (1 - g_X)(1 - g_Y)$$
$$= \left[\frac{1}{2} \langle f, f \rangle - \delta_X \cdot \delta_Y\right] + (\delta_X + 1 - g_X)(\delta_Y + 1 - g_Y)$$

Then it is enough to show that

$$\chi(f) \leq (\delta_X + 1 - g_X)(\delta_Y + 1 - g_Y).$$

Notice that it does not change $\delta_X \cdot \delta_Y - \frac{1}{2} \langle f, f \rangle$ if we add $a \times Y + X \times b$ to f. Hence, without loss of generality, we assume that $\delta_X > 2(g_X - 1)$, $\delta_Y > 2(g_Y - 1)$ and $h^0(\partial - f) = 0$. Therefore, since $h^1(f) \geq 0$, it suffices to show that

$$h^0(f) \leq (\delta_X + 1 - g_X)(\delta_Y + 1 - g_Y).$$

Let $r := \delta_Y + 1 - g_Y$. Take $y_1, \dots, y_r \in Y$ and put $X_{y_i} := X \times y_i$, $f_{y_i} := f \cap X_{y_i} \in \mathrm{Div} X_{y_i}$. Then we have $\deg f_{y_i} = \langle f, X_{y_i} \rangle = \delta_X > 2(g_X - 1)$ and $h^0(f_{y_i}) = \delta_X + 1 - g_X$ from the Riemann–Roch theorem for X_{y_i}. One have the map

$$H^0(V, f) \longrightarrow H^0(X_{y_1}, f_{y_1}) \oplus \cdots \oplus H^0(X_{y_r}, f_{y_r}) \tag{0.4}$$

given via the restriction $\varphi \mapsto (\varphi|_{X_{y_1}}, \dots, \varphi|_{X_{y_r}})$. Remark that $\dim H^0(V, f) = h^0(f)$ and $\dim H^0(X_{y_1}, f_{y_1}) \oplus \cdots \oplus H^0(X_{y_r}, f_{y_r}) = (\delta_X + 1 - g_X)(\delta_Y + 1 - g_Y)$. Therefore it is sufficient to show that the map (0.4) is injection for appropriate y_i's. Let φ be an element of the kernel of the above map. Note that $\mathrm{div}\varphi \geq -f + \sum_{i=1}^r X_{y_i}$. Let $\widetilde{x} \in X$ be a generic point and put $\widetilde{Y} := \widetilde{x} \times Y$. Then $\varphi|_{\widetilde{Y}} \in H^0\big(\widetilde{Y}, (f - \sum_{i=1}^r X_{y_i}) \cap \widetilde{Y}\big)$ with $X_{y_i} \cap \widetilde{Y} = (\widetilde{x}, y_i)$. Since we have $\deg f \cap \widetilde{Y} = \delta_Y > 2(g_Y - 1)$, by the Riemann–Roch theorem for $\widetilde{Y}$, we have $h^0(f \cap \widetilde{Y}) = \delta_Y + 1 - g_Y = r$. Choose the point y_i so that $h^0\big(f \cap \widetilde{Y} - \sum_{i=1}^n (\widetilde{x}, y_i)\big) = r - n$. Then, for $n = r$, we have $h^0\big(f \cap \widetilde{Y} - \sum_{i=1}^n (\widetilde{x}, y_i)\big) = 0$, whence $\varphi|_{\widetilde{Y}} = 0$. Therefore we conclude that $\varphi = 0$. This shows the desired claim, hence (0.3).

Grothendieck further analyzed the above proof of Mattuck–Tate ([Gr]). The map $h^0 : \mathrm{Div}(V) \to \mathbb{N}$ satisfies the following

(I) Riemann–Roch inequality : $h^0(f) + h^0(\partial - f) \geq \frac{1}{2}\langle f, f - \delta \rangle + \chi$.

(II) monotone : $h^0(f) > 0 \Longrightarrow h^0(f + g) \geq h^0(g)$.

(III) ampleness : $h^0(f) > 0 \Longrightarrow \langle f, H \rangle > 0$.

Here H is a hyperplane section, whence if $h^0(f) > 0$ we have $f \sim f + \mathrm{div}\varphi \geq 0$ and $\langle f, H \rangle = \deg(f + \mathrm{div}\varphi)|_H \geq 0$.

In our case, we view the function h^0 as $h^0 : C_c(g^{\mathbb{Z}}) \oplus \mathbb{Z}\delta_0 \oplus \mathbb{Z}\delta_\infty \to \mathbb{N}$, $\partial = (\text{const.})(\delta_0 + \delta_\infty)$ and $H = \delta_0 + \delta_\infty$.

The idea is as follows

$$\begin{aligned}\langle f, f \rangle > 0 &\overset{\text{(I)}}{\Longrightarrow} h^0(n \cdot f) \to \infty \ (n \to \infty) \quad (\text{after changing } f \text{ to } \pm f)\\ &\overset{\text{(III)}}{\Longrightarrow} \langle \pm f, H \rangle > 0.\end{aligned}$$

We have $\langle H, H \rangle > 0$. Suppose $\langle f, H \rangle = 0$. Then, by the above formula, we have $\langle f, f \rangle \leq 0$. Namely, $\langle \cdot, \cdot \rangle$ has the signature $(+, -, \dots, -)$ ("Hodge-index-theorem"). Taking the determinant of the symmetric matrix of the intersection numbers $\langle \cdot, \cdot \rangle$ of the vectors δ_0, δ_∞ and f, we have:

$$\begin{matrix} & & \delta_0 & \delta_\infty & f \\ \delta_0 & & & & \\ \delta_\infty & \det & & & \\ f & & & & \end{matrix}\qquad \det\begin{pmatrix} 0 & 1 & \delta_X \\ 1 & 0 & \delta_Y \\ \delta_X & \delta_Y & \langle f, f \rangle \end{pmatrix} \sim \det\begin{pmatrix} + & & \\ & - & \\ & & - \end{pmatrix} \geq 0$$

But the determinant is $2\delta_X\delta_Y - \langle f, f \rangle$, hence (0.3).

In [Har2], it was suggested that finding $h^0 : \mathcal{D} \to [0,\infty)$ where $\mathcal{D} \subseteq C_c^\infty(\mathbb{R}^+) \oplus \mathbb{C}\delta_0 \oplus \mathbb{C}\delta_\infty$ is sufficiently rich, satisfying (I), (II) and (III) with $\langle \cdot, \cdot \rangle$ defined by

$$\begin{aligned} \langle \delta_0, \delta_0 \rangle &= \langle \delta_\infty, \delta_\infty \rangle = 0, \quad \langle \delta_0, \delta_\infty \rangle = 1 \\ \langle f, \delta_0 \rangle &= \widehat{f}(0), \quad \langle f, \delta_\infty \rangle = \widehat{f}(1), \\ \langle f, g \rangle &= \langle f * g^\natural, \Delta \rangle , \end{aligned}$$

where

$$\langle f, \Delta \rangle := \widehat{f}(0) + \widehat{f}(1) - \sum_{\zeta_X(s)=0} \widehat{f}(s), \quad g^\natural(x) := x^{-1}\overline{g}(x^{-1}).$$

Then we get the Riemann hypothesis. This is the “Weil philosophy”. Remark that for Riemann hypothesis and the Artin conjecture for $L(s,\chi)$, we need to work with $\widetilde{X}$ a Galois covering of X associated with χ,

$$\chi : \mathrm{Gal}(\widetilde{X}/X) \longrightarrow \mathrm{GL}(V).$$

Thus we concentrate on zeta functions and do not deal with the L-functions.

0.6 Zeta Functions for Number Fields

Let K be a number field, $\mathcal{O}_K$ be the ring of integers of K, $K_\mathfrak{p}$ be the completion of K with respect to the prime ideal $\mathfrak{p}$ of K and $\mathcal{O}_\mathfrak{p}$ be the ring of integers of $K_\mathfrak{p}$. Let $\eta_1, \dots, \eta_{r_1+r_2}$ be the real primes, so we have

$$\bigoplus_{\mathfrak{p}|\eta} K_\mathfrak{p} = K \otimes_\mathbb{Q} \mathbb{R} = \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}.$$

Let $\mathbb{A}_K$ be the adele ring of K, $\mathbb{A}_K^*$ be the idele group of K and $\mathcal{O}_{\mathbb{A}_K}^* = \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^*$. Notice that for the real prime η, we have

$$\mathcal{O}_\eta^* = \begin{cases} \{\pm 1\} & \text{for } \mathbb{R}, \\ \mathbb{C}^{(1)} & \text{for } \mathbb{C}. \end{cases}$$

Then we have $\mathrm{Div}_K = \mathbb{A}_K^*/\mathcal{O}_\mathbb{A}^*$ and

$$\mathrm{Pic}_K = \mathbb{A}_K^*/\mathcal{O}_{\mathbb{A}_K}^* K^* \overset{|\cdot|_{\mathbb{A}_K}}{\longrightarrow} \mathbb{R}^+$$

with the kernel $\mathrm{Pic}_K^{(1)} = \mathbb{A}_K^{(1)}/\mathcal{O}_{\mathbb{A}_K}^* K^*$ being compact. Let $\boldsymbol{\mu}_K$ be the set of the root of unit in K. Then we have the exact sequence

$$* \longrightarrow \boldsymbol{\mu}_K \longrightarrow \mathcal{O}_K^* \longrightarrow \big(\prod_i K_{\eta_i}\big)^{(1)}/\prod_i \mathcal{O}_{\eta_i}^* \longrightarrow \mathrm{Pic}_K^{(1)} \longrightarrow \underbrace{\mathbb{A}_K^*/(\mathcal{O}_{\mathbb{A}_K}^* K^* \prod_i K_{\eta_i}^*)}_{\text{class } \mathcal{O}_K} \longrightarrow *$$

Since $\mathrm{Pic}_K^{(1)}$ is compact, we see that

$$\mathcal{O}_K^* \simeq \boldsymbol{\mu}_K \times \mathbb{Z}^{r_1+r_2-1}$$

and the ideal class group of $\mathcal{O}_K$ is finite.

The following normalization (it is different from the common Weil–Tate normalization) make sense: Let $\phi_{\mathcal{O}_\mathfrak{p}}$ be for finite $\mathfrak{p}$ the characteristic function of $\mathcal{O}_\mathfrak{p}$, and for real or complex η: $\phi_{\mathcal{O}_\eta}(x) = e^{-|x|_\eta^2}$. Let $dx_\mathfrak{p}$ be the additive Haar measure on $K_\mathfrak{p}$ normalized by $dx_\mathfrak{p}(\phi_{\mathcal{O}_\mathfrak{p}}) = 1$, that is,

$$dx_\eta = \begin{cases} \dfrac{dx}{\sqrt{\pi}} & \text{for } \mathbb{R}, \\ \dfrac{|dx \wedge d\overline{x}|_\eta}{2\pi} = \dfrac{dx_1}{\sqrt{\pi}}\dfrac{dx_2}{\sqrt{\pi}}, \; x = x_1 + ix_2 & \text{for } \mathbb{C}. \end{cases}$$

Let $||\cdot||_\mathfrak{p}$ be the absolute values on $K_\mathfrak{p}$ normalized by $d(ax_\mathfrak{p}) = ||a||_\mathfrak{p} dx_\mathfrak{p}$, that is,

$$||a||_\eta = \begin{cases} |a| & \text{for } \mathbb{R}, \\ |a|^2 & \text{for } \mathbb{C}. \end{cases}$$

Let $d^*a_\mathfrak{p}$ be the multiplicative Haar measure on $K_\mathfrak{p}^*$ normalized as

$$d^*a_\mathfrak{p} = \frac{da_\mathfrak{p}}{||a||_\mathfrak{p}} \cdot \begin{cases} \left(1 - \mathbb{N}\mathfrak{p}^{-1}\right)^{-1} & \text{for } \mathfrak{p} \nmid \eta, \\ \sqrt{\pi} & \text{for } \mathbb{R}, \\ 1 & \text{for } \mathbb{C}. \end{cases}$$

Let $\psi_\mathfrak{p} : K_\mathfrak{p} \to \mathbb{C}^{(1)}$ be the additive character normalized (modulo $\mathcal{O}_\mathfrak{p}^*$) by $\mathcal{F}\phi_{\mathcal{O}_\mathfrak{p}} = \phi_{\mathcal{O}_\mathfrak{p}}$, that is, $\ker \psi_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ for finite $\mathfrak{p}$, and

$$\psi_\eta(x) = \begin{cases} e^{2ix} & \text{for } \mathbb{R}, \\ e^{i(x+\overline{x})} & \text{for } \mathbb{C}. \end{cases}$$

Here

$$\mathcal{F}_\mathfrak{p}\varphi(y) = \int_{K_\mathfrak{p}} \varphi(x)\psi_\mathfrak{p}(xy)dx_\mathfrak{p}.$$

Similarly, we put $\psi_{\mathbb{A}_K} = \prod_\mathfrak{p} \psi_\mathfrak{p} : \mathbb{A}_K \to \mathbb{C}^{(1)}$ and $\mathcal{F} = \bigotimes_\mathfrak{p} \mathcal{F}_\mathfrak{p}$. Take $\partial \in \mathbb{A}_K^*/\mathcal{O}_{\mathbb{A}_K}^* K^*$ (= the canonical class) such that $\psi_{\mathbb{A}_K}(\partial K) = 1$. Namely, if $\partial_\mathfrak{p}^{-1}\mathcal{O}_\mathfrak{p}$ is the usual different of $K_\mathfrak{p}/\mathbb{Q}_p$ for finite p's, then

$$\partial_\eta = \begin{cases} \pi & \text{for } \mathbb{R}, \\ 2\pi & \text{for } \mathbb{C}. \end{cases}$$

We define the local zeta function by

$$\zeta_\mathfrak{p}(s) := \int_{K_\mathfrak{p}^*} \phi_{\mathcal{O}_\mathfrak{p}}(a)||a||_\mathfrak{p}^s d^*a = \begin{cases} \left(1 - \mathbb{N}\mathfrak{p}^{-s}\right)^{-1} & \text{for } \mathfrak{p} \nmid \eta, \\ \varGamma\left(\frac{s}{2}\right) & \text{for } \mathbb{R}, \\ \varGamma(s) & \text{for } \mathbb{C}. \end{cases}$$

and the global zeta function by

$$\zeta_K(s) := \prod_{\mathfrak{p}} \zeta_{\mathfrak{p}}(s) = \int_{\mathbb{A}^*} \phi_{\mathcal{O}_{\mathbb{A}_K}}(a)||a||_{\mathbb{A}_K}^s d^*a$$
$$= \int_{\mathrm{Pic}_K = \mathbb{A}_K^*/\mathcal{O}_{\mathbb{A}_K}^* K^*} \Big(\sum_{\gamma \in K^*} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma a)\Big)||a||_{\mathbb{A}_K}^s d^*a,$$

where $\phi_{\mathcal{O}_{\mathbb{A}_K}} := \prod_{\mathfrak{p}} \phi_{\mathcal{O}_{\mathfrak{p}}}$. Then the Riemann–Roch theorem asserts that

$$dx(\mathbb{A}_K/K) \cdot \sum_{\gamma \in K} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma a) = ||a||_{\mathbb{A}_K}^{-1} \sum_{\gamma} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma \partial a^{-1}).$$

The proof is the same as the one for (0.1). Taking log, we have

$$h^0(a) = \log\Big[\sum_{\gamma \in (a^{-1})_{\mathrm{fin}}} e^{-\sum_i |a_{\eta_i}\gamma|_{\eta_i}^2}\Big] = \deg a + h^0(\partial a^{-1}) + \chi.$$

Note that $\deg a = \log ||a||_{\mathbb{A}}^{-1}$ and $\chi = -\log dx(\mathbb{A}_K/K) = \frac{1}{2}\log ||\partial||_{\mathbb{A}_K}$ with

$$||\partial||_{\mathbb{A}_K}^{-1} = \frac{|D_K|}{\pi^{r_1}(2\pi)^{2r_2}}$$

where D_K is the discriminant of K. We have

$$\zeta_K(s) = \int_{||a||_{\mathbb{A}_K} \le 1} \sum_{\gamma \in K} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma a)||a||_{\mathbb{A}_K}^s d^*a - \frac{H_K}{s} + \int_{||a|| \ge 1} \sum_{\gamma \in K^*} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma a)||a||_{\mathbb{A}_K}^s d^*a.$$

(Here $H_K =$ "$d^*a(\mathrm{Pic}_K^{(1)})$" is obtained by comparing the Haar measure d^*a and the measure $d^\circ a \otimes \frac{dt}{t}$ on $\mathrm{Pic}_K = \mathrm{Pic}_K^{(1)} \times \mathbb{R}^+$, via $d^*a = H_K \cdot d^\circ a \otimes \frac{dt}{t}$.) Since the Riemann–Roch theorem gives

$$\int_{||a||_{\mathbb{A}_K} \le 1} \sum_{\gamma \in K} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma a)||a||_{\mathbb{A}_K}^s d^*a$$
$$= dx(\mathbb{A}_K/K)^{-1} \int_{||a||_{\mathbb{A}_K} \le 1} \sum_{\gamma \in K} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma \partial a^{-1})||a||_{\mathbb{A}_K}^{s-1} d^*a$$
$$= dx(\mathbb{A}_K/K)^{-1} \int_{||a||_{\mathbb{A}_K} \ge 1} \sum_{\gamma \in K^*} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma \partial a)||a||_{\mathbb{A}_K}^{1-s} d^*a + dx(\mathbb{A}_K/K)^{-1}\frac{H_K}{s-1},$$

we have

$$\zeta_K(s) = dx(\mathbb{A}_K/K)^{-1} \int_{||a||_{\mathbb{A}_K} \ge 1} \sum_{\gamma \in K^*} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma \partial a)||a||_{\mathbb{A}_K}^{1-s} d^*a$$
$$+ \int_{||a|| \ge 1} \sum_{\gamma \in K} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\gamma a)||a||_{\mathbb{A}_K}^s d^*a + dx(\mathbb{A}_K/K)^{-1}\frac{H_K}{s-1} - \frac{H_K}{s}.$$

This gives the meromorphic continuation of $\zeta_K(s)$ to the whole plane except for simple poles at $s = 0, 1$ with residues

$$\operatorname*{Res}_{s=0}\zeta_K(s) = -H_K, \qquad \operatorname*{Res}_{s=1}\zeta_K(s) = H_K \cdot ||\partial||_{\mathbb{A}_K}^{-\frac{1}{2}}.$$

Here we have the equality

$$H_K = \text{“}d^*a(\mathrm{Pic}_K^{(1)})\text{”} = \frac{2^{r_1} h_K R_K}{\#\boldsymbol{\mu}_K},$$

where R_K is the regulator of K and h_K is the class number of K. Then one obtains the functional equation

$$\zeta_K(s) = ||\partial||_{\mathbb{A}_K}^{s-\frac{1}{2}} \zeta_K(1-s).$$

For the proof of the functional equation, we go to the Weil–Tate notations: Let $\widetilde{\psi}(x) := \psi(\partial x)$. Then $\widetilde{\psi}$ is trivial on K. Let $\widetilde{dx} := dx(\mathbb{A}_K/K)^{-1} \cdot dx$ so that $\widetilde{dx}(\mathbb{A}_K/K) = 1$. Let $\widetilde{\mathcal{F}}$ be the Fourier transform with respect to $\widetilde{\psi}$:

$$\widetilde{\mathcal{F}}\varphi(y) := \int_{\mathbb{A}} \varphi(x)\widetilde{\psi}(xy)\widetilde{dx}.$$

We see that $\widetilde{\mathcal{F}}$ is the self-dual Fourier transform: $\widetilde{\mathcal{F}}\widetilde{\mathcal{F}}\phi_{\mathcal{O}_{\mathbb{A}_K}} = \phi_{\mathcal{O}_{\mathbb{A}_K}}$. We have $\widetilde{\mathcal{F}}\phi_{\mathcal{O}_{\mathbb{A}_K}}(y) = dx(\mathbb{A}_K/K)^{-1}\phi_{\mathcal{O}_{\mathbb{A}_K}}(\partial x)$ and get

$$\zeta_K(\phi_{\mathcal{O}_{\mathbb{A}_K}}(x), s) = \zeta_K(\widetilde{\mathcal{F}}\phi_{\mathcal{O}_{\mathbb{A}_K}}, 1-s).$$

Hence we have

$$\zeta_K(s) = dx(\mathbb{A}_K/K)^{-1} \prod_{\mathfrak{p}} \int_{\mathbb{A}_K} \phi_{\mathcal{O}_{\mathbb{A}_K}}(\partial a)||a||_{\mathbb{A}_K}^{1-s} d^*a = dx(\mathbb{A}_K/K)^{-1}||\partial||_{\mathbb{A}_K}^{s-1}\zeta_K(1-s).$$

Note that, taking $s = \frac{1}{2}$, we have $dx(\mathbb{A}_K/K) = ||\partial||_{\mathbb{A}_K}^{-\frac{1}{2}}$.

For Weil–Tate's case, $\widetilde{\psi} = \psi_{\text{cann.}/\mathbb{Q}} \circ \mathrm{Tr}_{K/\mathbb{Q}}$ and $\psi_{\text{cann.}/\mathbb{Q}}$ is the “canonical” character of $\mathbb{Q}$. Note that for number fields K we have a canonical embedding: $\mathbb{Q} \hookrightarrow K$, but for function fields K there is no canonical embedding: $k(t) \hookrightarrow K$; a choice of such embedding is equivalent to a choice of $t \in K^* \backslash k^*$ and this is equivalent to $t : X \to \mathbb{P}^1(K)$. Then, for the number field, we have the “canonical” differential form which is the pull back of the form from $\mathbb{Q}$, but not for function fields. Also the normalization $\widetilde{dx}(\mathbb{A}_K/K) = 1$ only fixes the global measure $\widetilde{dx}$ but not the local factor $\widetilde{dx_{\mathfrak{p}}}$, while the normalization $dx_{\mathfrak{p}}(\phi_{\mathcal{O}_{\mathfrak{p}}}) = 1$ fixes the local factors.

These normalizations are nicer than Weil and Tate's: In the Weil and Tate's normalization, we have

$$\zeta_\eta(s) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) & \text{for } \mathbb{R}, \\ (2\pi)^{1-s}\Gamma(s) & \text{for } \mathbb{C}, \end{cases} \quad d^*x = \begin{cases} \dfrac{dx}{|x|} & \text{for } \mathbb{R}, \\ \dfrac{|dx \wedge d\overline{x}|}{|x|^2} & \text{for } \mathbb{C}, \end{cases} \quad \phi_{\mathcal{O}_\eta} = \begin{cases} e^{-\pi x^2} & \text{for } \mathbb{R}, \\ e^{-2\pi|x|^2} & \text{for } \mathbb{C}. \end{cases}$$

Hence the functional equation for the global zeta function $\zeta_{\mathbb{A}_K}(s)$ is given by

$$\zeta_{\mathbb{A}_K}(s) := \prod_{\mathfrak{p}} \zeta_{\mathfrak{p}}(s) = |D_K|^{\frac{1}{2}-s}\zeta_{\mathbb{A}_K}(1-s)$$

with residues at $s = 0, 1$

$$\operatorname*{Res}_{s=0}\zeta_{\mathbb{A}_K}(s) = -C_K, \qquad \operatorname*{Res}_{s=1}\zeta_{\mathbb{A}_K}(s) = C_K|D_K|^{-\frac{1}{2}},$$

where $C_K := 2^{r_1}(2\pi)^{r_2}h_K R_K/\#\boldsymbol{\mu}_K$. Further, our normalization is more natural from the "q-point of view", which will apear below, and the η-limit which relates the quantum and the reals. Namely, we can consider in the η-limit ⓝ $\frac{q^n}{1-q} \to |x^2|$, rather than $\pi|x|^2$, and we get polynomials in $|x|^2$ rather than $\pi|x|^2$, etc. Nevertheless, Tate and Weil's normalization are so popular that we will adopt their notations in the text.

This method also work for L-function. Namely, one can obtain the analytic continuation and functional equation of $L(s,\chi)$ where χ is a character of $\mathbb{A}_K^*/K^*$. Here, the point is that we consider only the characters which are trivial on K^*. For such χ's, we see that $||a||_{\mathbb{A}_K}^s\chi(a)\frac{d^*a}{L(s,\chi)} \in \mathcal{S}^*(\mathbb{A})^{K^*}$, the space of K^*-invariant distributions on $\mathbb{A}$, and is analytic in s. Namely, we are studying the action of $\mathbb{A}_K^*/K^*$ on "$\mathbb{A}_K/K^*$" which is a problematic space! When we concentrate on the zeta function, we study the action of $\mathbb{A}_K^*/\mathcal{O}_{\mathbb{A}_K}^*K^*$ on $\mathbb{A}_K/\mathcal{O}_{\mathbb{A}_K}^*K^*$. Note that the Tate distribution $\tau_{\mathbb{A}_K}^s = ||a||_{\mathbb{A}_K}^s\frac{d^*a}{\zeta_K(s)} \in \mathcal{S}^*(\mathbb{A}_K)^{\mathcal{O}_{\mathbb{A}_K}^*K^*}$ is holomorphic in s. For $\mathrm{Re}(s) > 1$, $\tau_{\mathbb{A}_K}^s$ has support at $\mathbb{A}_K^*$, whence there is no problem. The problem occurs for $\mathrm{Re}(s) \le 1$, e.g., $\tau_{\mathbb{A}_K}^1 = dx$, the additive Haar measure on $\mathbb{A}_K$ and $dx(\mathbb{A}_K^*) = 0$ (this is equivalent to $\zeta_{\mathbb{A}_K}(1) = \infty$). Notice that while $\mathcal{S}^*(\mathbb{A}_K)^{K^*}$ is very rich, $\mathcal{S}(\mathbb{A}_K)^{K^*}$ is empty. Therefore we have to work with the non-commutative algebra $\mathcal{S}(\mathbb{A}) \rtimes K^*$ or for zeta function $\mathcal{S}(\mathbb{A})^{\mathcal{O}_{\mathbb{A}_K}^*} \rtimes K^*/\boldsymbol{\mu}_K$. Note that, for $K = \mathbb{Q}$, Tate's proof reduces to Riemann's but Tate's $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ is better than Riemann's $\mathbb{R}/\mathbb{Z} = \mathbb{A}_{\mathbb{Q}}/\widehat{\mathbb{Z}}\mathbb{Q}$ because it carry the ergodic $\mathbb{Q}^*$ action (vs $\mathbb{Z}^* = \{\pm 1\}$ action on $\mathbb{R}/\mathbb{Z}$).

0.7 Weil's Explicit Sum Formula

Normalize the critical line $\mathrm{Re}(s) = \frac{1}{2}$ to $i\mathbb{R}$. Namely, we consider the function $\zeta_{\mathbb{A}_K}(s+\frac{1}{2})$, and shift $f(x) := x^{-\frac{1}{2}}f(x)$ so that $f^{\natural}(x) := f^*(x) = \overline{f}(x^{-1})$. Then we have

$$\begin{aligned} \text{“}\langle f, \Delta\rangle\text{”} &= \widehat{f}(\frac{1}{2}) + \widehat{f}(-\frac{1}{2}) - \sum_{\zeta_{\mathbb{A}_K}(s+\frac{1}{2})=0} \widehat{f}(s) \\ &= -\frac{1}{2\pi i}\oint \widehat{f}(s)d\log\zeta_{\mathbb{A}_K}(\frac{1}{2}+s) \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\widehat{f}(-s)-\widehat{f}(s)\right]d\log\zeta_{\mathbb{A}_K}\big(\frac{1}{2}+s\big)+f(1)\log||\partial||_{\mathbb{A}_K}\quad (c>\frac{1}{2})$$

(∵ functional equation),

$$= -\sum_{\mathfrak{p}}\frac{1}{2\pi i}\widehat{f}(s)\int_{-i\infty}^{i\infty}d\log\frac{\zeta_{\mathfrak{p}}(\frac{1}{2}+s)}{\zeta_{\mathfrak{p}}(\frac{1}{2}-s)||\partial_{\mathfrak{p}}||_{\mathbb{A}}}\qquad (\because \text{ Euler product})$$

$$= \sum_{\mathfrak{p}}\text{“}\langle f,\Delta\rangle_{\mathfrak{p}}\text{”}.$$

For finite $\mathfrak{p}$, we have

$$\langle f,\Delta\rangle_{\mathfrak{p}} = \log\mathbb{N}\mathfrak{p}\cdot\sum_{n\neq 0}(\mathbb{N}\mathfrak{p})^{-\frac{|n|}{2}}f(\mathbb{N}\mathfrak{p}^n)+f(1)\cdot\log||\partial_{\mathfrak{p}}||_{\mathbb{A}_K}.$$

Note that $\log||\partial_{\mathfrak{p}}||_{\mathbb{A}_K} = \langle\Delta,\Delta\rangle_{\mathfrak{p}}$. On the other hand, for $\mathfrak{p}|\eta$, it is more complicated but we give the finite form of $\langle f,\Delta\rangle_{\mathfrak{p}}$ in [Har1], [Har2] and [Har5].

The following formula connects the “Weil philosophy” with “Tate philosophy” and seems fundamental (see [Har1] and [Har2]): For all $\mathfrak{p}$, we have

$$\begin{aligned}\langle f,\Delta\rangle_{\mathfrak{p}} &= \mathcal{F}_{\mathfrak{p}}\log|x|_{\mathfrak{p}}^{-1}\mathcal{F}_{\mathfrak{p}}^{-1}|x|_{\mathfrak{p}}^{-\frac{1}{2}}(f|_{\mathfrak{p}})(1)\\ &= \frac{\partial}{\partial s}\Big|_{s=0}\mathcal{F}_{\mathfrak{p}}|x|_{\mathfrak{p}}^{-s}\mathcal{F}_{\mathfrak{p}}^{-1}|x|_{\mathfrak{p}}^{-\frac{1}{2}}(f|_{\mathfrak{p}})(1)\\ &= \frac{\partial}{\partial s}\Big|_{s=0}R_{\mathfrak{p}}^{s}|x|_{\mathfrak{p}}^{-\frac{1}{2}}f|_{\mathfrak{p}}(1).\end{aligned}\tag{0.5}$$

Here we denote by $f|_{\mathfrak{p}}(x) := f(|x|_{\mathfrak{p}})$, the composition of f with $|\cdot|_{\mathfrak{p}}$, and

$$R_{\mathfrak{p}}^{s} := \mathcal{F}_{\mathfrak{p}}\big(|x|_{\mathfrak{p}}^{-s}\big) = \zeta_{\mathfrak{p}}(1-s)\frac{|x|_{\mathfrak{p}}^{s-1}dx_{\mathfrak{p}}}{\zeta_{\mathfrak{p}}(s)}$$

by Tate's (local) functional equation. Then $R_{\mathfrak{p}}^{s}$ is the Riesz potential. This formula was checked in [Har1] by direct calculation and reproved by later authors, cf. [Bur2], and [C2] where it is reproved in Appendix 2 in the asymptotic form:

$$\langle f,\Delta\rangle_{\mathfrak{p}} = \mathcal{F}_{\mathfrak{p}}B_c(x)\log|x|_{\mathfrak{p}}^{-1}\mathcal{F}_{\mathfrak{p}}^{-1}(f|_{\mathfrak{p}})(1)+o(c)\qquad (c\to\infty),$$

where

$$B_c(x) = \begin{cases}1 & \text{for } |x|_{\mathfrak{p}}\le c,\\ 0 & \text{for } |x|_{\mathfrak{p}}> c.\end{cases}$$

We like the Riesz potential formulation because

$$\sum_{\mathfrak{p}}\frac{\partial}{\partial s}\Big|_{s=0}R_{\mathfrak{p}}^{s} = \frac{\partial}{\partial s}\Big|_{s=0}\bigotimes_{\mathfrak{p}}R_{\mathfrak{p}}^{s},$$

which change sum over primes (it is rarely understood) to product over primes. We also can write it as (normalized) trace (cf. [Har2]), indeed, in more than one way (cf [Har5]). These trace formulas can be globalized to a form that "shows" why the Weil distribution is positive (but does not prove positivity because they are obtained via analytic continuation): The Riemann hypothesis for $\mathbb{Q}$ is equivalent to the positivity

$$\sum_{q\in\mathbb{Q}^*} \underset{s=0}{\mathrm{CT}}\mathrm{Tr}\Big(e^{-s\sum_{\mathfrak{p}} N_{\mathfrak{p}}}\pi(f)\pi(f)^*\pi(q)\Big) \geq 0,$$

where $N_{\mathfrak{p}}$ is the number operator associated with the Laguerre basis at $\beta = 1$, or the Jacobi basis at $\alpha = \beta = 1$ (cf. [Har5] (13.5.70) and (13.5.71)) and f is a function such that $\widehat{f}(\frac{1}{2}) + \widehat{f}(-\frac{1}{2}) = 0$. A. Connes also tried to globalize the asymptotic form of the formula and got

$$\mathrm{Tr}(P_c(\pi(f))) = 2(\log c)\cdot f(1) + \sum_{\mathfrak{p}} \langle f, \Delta\rangle_{\mathfrak{p}} + o(1) \qquad (c\to\infty),$$

where P_c is the projection approximating "$\mathcal{F}_{\mathfrak{p}} B_c \mathcal{F}_{\mathfrak{p}}^{-1} B_c$", but this global formula is equivalent to Riemann hypothesis. Recently, Meyer [Mey2] rewrote the formula as a cyclic trace.

The following is Burnol's proof of our formula (0.5): First we consider f on $\mathbb{Q}_p^*/\mathbb{Z}_p^*$. We have the commutative diagram of spaces and isomorphisms:

$$\begin{array}{ccc}
L^2\big(\mathbb{Q}_p^*, d^*x_{\mathfrak{p}}\big)^{\mathbb{Z}_p^*} & \underset{\cdot|x|_p^{\frac{1}{2}}}{\overset{\cdot|x|_p^{-\frac{1}{2}}}{\rightleftarrows}} & L^2\big(\mathbb{Q}_p, dx_p\big)^{\mathbb{Z}_p^*} \\
\Big\downarrow {\scriptstyle |x|_p^s d^*x_p} \wr & & \wr\Big\downarrow {\scriptstyle |x|_{\mathfrak{p}}^{\frac{1}{2}+s}\frac{d^*x_{\mathfrak{p}}}{\zeta_p(\frac{1}{2}+s)}} \\
L^2(i\mathbb{R}/\lambda_p, d^\circ s) & \underset{\zeta_p(\frac{1}{2}+s)\cdot}{\overset{\cdot\zeta_p(\frac{1}{2}+s)^{-1}}{\rightleftarrows}} & L^2\big(i\mathbb{R}/\lambda_p, |\zeta_p(\frac{1}{2}+s)|^s d^\circ s\big)
\end{array}$$

(with $\lambda_p = \frac{2\pi i}{\log p}\mathbb{Z}$ for finite p's). On each space, we have the following corresponding operators:

On $L^2\big(\mathbb{Q}_p^*, d^*x_p\big)^{\mathbb{Z}_p^*}$:

$\mathcal{F}$: (i) $|x|_p^{\frac{1}{2}}\mathcal{F}_p|x|_p^{-\frac{1}{2}}$

I : (ii) $x \mapsto x^{-1}$

$\mathit{\Gamma}$: (iii) $\mathit{\Gamma} = \mathcal{F}\cdot I$

(o) $\cdot\log|x|_p$

$(*)$ $f*$

$(\star)$ $\log|x|_p + |x|^{\frac{1}{2}}\mathcal{F}_p\log|x|_p\mathcal{F}_p^{-1}|x|_p^{-\frac{1}{2}}$

On $L^2\big(\mathbb{Q}_p, dx_p\big)^{\mathbb{Z}_p^*}$:

$\mathcal{F}$: (i) $\mathcal{F}_p$

I : (ii) $\varphi(x) \mapsto \dfrac{1}{|x|_p}\varphi\big(\dfrac{1}{x}\big)$

$\mathit{\Gamma}$: (iii) $\mathit{\Gamma} = \mathcal{F}\cdot I$

(o) $\log|x|_p$

$(*)$ $\pi^1(f)$

$(\star)$ $\big(\log|x|_p + \mathcal{F}_p\log|x|_p\mathcal{F}_p^{-1}\big)$

On $L^2(i\mathbb{R}/\lambda_p, d^\circ s)$:

$\mathcal{F}$: (i) $\widehat{f}(s) \mapsto \widehat{f}(-s)\dfrac{\zeta_p(\frac{1}{2}+s)}{\zeta_p(\frac{1}{2}-s)}$

I : (ii) $s \mapsto -s$

Γ : (iii) $\cdot \dfrac{\zeta_p(\frac{1}{2}+s)}{\zeta_p(\frac{1}{2}-s)}$

(o) $\dfrac{\partial}{\partial s}$

(∗) $\cdot \widehat{f}(s)$

(⋆) $\dfrac{\partial}{\partial s} - \Gamma\dfrac{\partial}{\partial s}\Gamma^{-1} = \left(d\log\dfrac{\zeta_p(\frac{1}{2}+s)}{\zeta_p(\frac{1}{2}-s)}\right)\cdot$

On $L^2\big(i\mathbb{R}/\lambda_p, |\zeta_{\mathfrak{p}}(\frac{1}{2}+s)|^s d^\circ s\big)$:

$\mathcal{F}$: (i) $s \mapsto -s$

I : (ii) $\widehat{f}(s) \mapsto \widehat{f}(-s)\dfrac{\zeta_p(\frac{1}{2}+s)}{\zeta_p(\frac{1}{2}-s)}$

Γ : (iii) $\cdot \dfrac{\zeta_p(\frac{1}{2}+s)}{\zeta_p(\frac{1}{2}-s)}$

(o) $\dfrac{\partial}{\partial s} + d\log\zeta_p(\dfrac{1}{2}+s)$

(∗) $\cdot \widehat{f}(s)$

(⋆) $\left(d\log\dfrac{\zeta_p(\frac{1}{2}+s)}{\zeta_p(\frac{1}{2}-s)}\right)\cdot$

Then we have

$$\begin{aligned}
&\operatorname{Tr}\big((\log|x|_p + \mathcal{F}_p\log|x|_p\mathcal{F}^{-1})\pi^1(f)\big)\\
&\quad= \operatorname{Tr}\Big(\big(\frac{\partial}{\partial s} - \Gamma\frac{\partial}{\partial s}\Gamma^{-1}\big)\cdot\widehat{f}(s)\Big)\\
&\quad= \int_{(\mathbb{Q}_p^*/\mathbb{Z}_p^*)^\wedge}\widehat{f}(s)d\log\frac{\zeta_p(\frac{1}{2}+s)}{\zeta_p(\frac{1}{2}-s)} = -\langle f,\Delta\rangle_p\\
&\quad= \big(\log|x|_p - \Gamma\log|x|_p\Gamma^{-1}\big)f(1)\\
&\quad= \big(\log|x|_p + |x|_p^{\frac{1}{2}}\mathcal{F}_p|x|_p^{-\frac{1}{2}}\log|x|_p|x|_p^{\frac{1}{2}}\mathcal{F}_p^{-1}|x|_p^{-\frac{1}{2}}\big)f(1)\\
&\quad= \mathcal{F}_p\log|x|_p\mathcal{F}_p^{-1}|x|_p^{-\frac{1}{2}}f(1).
\end{aligned}$$

We change from f on $\mathbb{Q}_p^*/\mathbb{Z}_p^*$ to f on $\mathbb{R}^+$. Since we have the injection $\mathbb{Q}_p^*/\mathbb{Z}_p^* \to \mathbb{R}^+$ via $|\cdot|_p$, by duality we have the surjection

$$\pi : i\mathbb{R} \longrightarrow \big(\mathbb{Q}_p^*/\mathbb{Z}_p^*\big)^\wedge,$$

and the local and global Mellin transforms

$$M_p : \mathcal{S}(\mathbb{Q}_p^*/\mathbb{Z}_p^*) \xrightarrow{\sim} \mathcal{S}(d^\circ s_p), \qquad M : \mathcal{S}(\mathbb{R}^+) \xrightarrow{\sim} \mathcal{S}(i\mathbb{R})$$

are related by $\pi_* Mf = M_p(f\circ|\cdot|_p) = M_p(f|_p)$ so that

$$\begin{aligned}
\big(\widehat{f}(s), \widehat{\nu_p}\circ\pi_p\big)_{i\mathbb{R}} = \big(\pi_*\widehat{f}(s), \widehat{\nu_p}\big)_{(\mathbb{Q}_p^*/\mathbb{Z}_p^*)^\wedge} &= \big(M_p^{-1}\pi_* Mf, \nu_p\big)_{\mathbb{Q}_p^*/\mathbb{Z}_p^*}\\
&= \big(f\circ|\cdot|_p, \nu_p\big)_{\mathbb{Q}_p^*/\mathbb{Z}_p^*}.
\end{aligned}$$

This shows that

$$\begin{aligned}
-\operatorname{Tr}\big((\log|x|_p + \mathcal{F}_p\log|x|_p\mathcal{F}_p^{-1})\pi_p^1(f|_p)\big) &= \mathcal{F}_p\log|x|_p^{-1}\mathcal{F}_p^{-1}|x|_p^{-\frac{1}{2}}(f|_p)(1)\\
&= -\int_{i\mathbb{R}}\widehat{f}(s)d\log\frac{\zeta_p(\frac{1}{2}+s)}{\zeta_p(\frac{1}{2}-s)}\\
&= \langle f,\Delta\rangle_p\,.
\end{aligned}$$

Hence we obtain (0.5).

Infinitesimal neighborhoods of the diagonal $X = \mathrm{Spec}\mathbb{Z}$ in the (non-existing, but see [Har6]) surface $X \times X$ is the tangent space TX. Moreover, on X, we have $\Delta^* T(X \times X) =: V$, a rank 2 vector bundle on X. This explains the connection of the Riemann hypothesis with SL_2. The "phase-space" V has the symplectic form $\langle \cdot, \cdot \rangle$. Its "meromorphic sections" are $V(\mathbb{Q}) = \mathbb{Q} \times \mathbb{Q}$, its adelic section are $V(\mathbb{A}) = \mathbb{A} \times \mathbb{A}$ and its "holomorphic sections" at p are $V(\mathbb{Z}_p) = \mathbb{Z}_p \times \mathbb{Z}_p = \big\{(x,y) \in V(\mathbb{Q}_p) \,\big|\, |x,y|_{\mathfrak{p}} \leq 1\big\}$ with 2-dimensional absolute value

$$|x,y|_{\mathfrak{p}} := \begin{cases} \max\{|x|_{\mathfrak{p}}, |y|_{\mathfrak{p}}\} & \text{for } \mathfrak{p} \nmid \eta, \\ \sqrt{|x|_{\mathfrak{p}}^2 + |y|_{\mathfrak{p}}^2} & \text{for } \mathfrak{p} | \eta. \end{cases}$$

For $a \in \mathbb{A}_K^*$ (resp. $a \in \mathbb{Q}_p^*$), the "trace" of the Frobenius $\mathfrak{f}_a$ on V is the line $(1 : a) \in \mathbb{P}^1(V)$. Note that any line $\ell \in \mathbb{P}^1(V)$ gives a maximal abelian subgroup $\ell \times \mathbb{A}^1$ of the Heisenberg group $\mathrm{Heis} = V \times \mathbb{A}^1$, hence the representation of Heis given by $\mathrm{Ind}_{\ell \times \mathbb{A}^1}^{\mathrm{Heis}}(\psi)$ is the fundamental irreducible representation of Heis with central character ψ. Therefore in any realization of this representation we have a (unique up to constant multiple) distribution δ_ℓ which is ψ-invariant under $\ell \times \mathbb{A}^1$, i.e., ℓ-invariant. For example, taking the Schrödinger model we get an embedding $\mathbb{P}^1(\mathbb{A}) \hookrightarrow \mathbb{P}^1(\mathcal{S}^*(\mathbb{A}))$, $\ell \mapsto \delta_\ell$; e.g., for $\varphi \in \mathcal{S}(\mathbb{A})$ (resp. $\mathcal{S}(\mathbb{Q}_p)$), we have the distribution δ_0 and δ_∞:

$$\delta_0(\varphi) = \varphi(0) = \tau^0(\varphi) \qquad \text{(unique invariant with respect to } \varphi(x) \mapsto \varphi(x)\psi(yx)),$$
$$\delta_\infty(\varphi) = \int_{\mathbb{A}_K} \varphi(x)dx = \tau^1(\varphi) \qquad \text{(unique invariant with respect to } \varphi(x) \mapsto \varphi(x)\psi(x+y)).$$

The unitary action of the multiplicative group on these distributions is

$$\pi^1(f)(\tau^0) = \widehat{f}\big(-\frac{1}{2}\big)\tau^0, \qquad \pi^1(f)(\tau^1) = \widehat{f}\big(\frac{1}{2}\big)\tau^1.$$

For the Heis story, see Chap. 12 of [Har5]. For the (many) trace formulation of the Weil distribution and approximations of the Riemann hypothesis using q, see Chap. 13 of [Har5]. For $X = \overline{\mathrm{Spec}\mathcal{O}_K}$, the compactification of $\mathrm{Spec}\mathcal{O}_K$ (e.g., $\overline{\mathrm{Spec}\mathbb{Z}} = \mathrm{Spec}\mathbb{Z} \cup \{\eta\}$) as a true geometrical object, see [Har6]. You will find in [Har6] also the (compactified) surface $X \times_{\mathbb{F}} X$ with $\mathbb{F}$ being the "field with one element".

1

Gamma and Beta Measures

Summary. In Sect. 1.1 we start from $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$, the inverse limit of $\mathbb{Z}/p^n$. It defines a rooted tree with valencies p. The collection of all paths of this tree is $\mathbb{Z}_p$. Similarly we have a tree for projective space $\varprojlim \mathbb{P}^1(\mathbb{Z}/p^n) = \mathbb{P}^1(\mathbb{Z}_p) = \mathbb{P}^1(\mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$, the inverse limit of the projective space $\mathbb{P}^1(\mathbb{Z}/p^n)$. This is the $p+1$-regular tree. The boundary is the collection of all path on the tree and is identified with the projective line $\mathbb{P}^1(\mathbb{Q}_p)$.

In Sect. 1.2 we put the p-adic γ-measure $\tau^{\beta}_{\mathbb{Z}_p}$ on $\mathbb{Z}_p$ defined by

$$\tau^{\beta}_{\mathbb{Z}_p}(x) = |x|^{\beta}_p \frac{d^* x}{\zeta_p(\beta)} \qquad (\beta > 0)$$

and we give the real analogue, the usual γ-measure

$$\tau^{\beta}_{\mathbb{Z}_\eta}(x) = \phi_{\mathbb{Z}_\eta}(x)|x|^{\beta}_\eta \frac{d^* x}{\zeta_\eta(\beta)} \qquad (\beta > 0),$$

where $\phi_{\mathbb{Z}_\eta}(x) = e^{-\pi x^2}$ is the "characteristic function of $\mathbb{Z}_\eta$".

In Sect. 1.3 we similarly put on $\mathbb{P}^1(\mathbb{Q}_p)$ the measure

$$\mathrm{pr}_*\big(\tau^{\alpha}_{\mathbb{Z}_p}(x) \otimes \tau^{\beta}_{\mathbb{Z}_p}(x)\big).$$

This is a projection of the probability measure on the plain $\mathbb{Q}_p \times \mathbb{Q}_p$ down to the projective line $\mathbb{P}^1(\mathbb{Q}_p)$. We denote it by $\tau^{\alpha,\beta}_p$ and call it the β-measure; it is given in terms of the canonical distance function ρ on $\mathbb{P}^1(\mathbb{Q}_p)$:

$$\tau^{\alpha,\beta}_p(x) = \rho^{\alpha}_{\infty}(x)\rho^{\beta}_0(x)\frac{d^* x}{\zeta_p(\alpha,\beta)}.$$

Here $\zeta_p(\alpha,\beta) = \zeta_p(\alpha)\zeta_p(\beta)/\zeta_p(\alpha+\beta)$ is the beta function. Again we obtain the Markov chain on this tree.

We notice that the measure $\tau^{\beta}_{\mathbb{Z}_p}$ is $\mathbb{Z}^*_p$-invariant, whence we can look at it as a probability measure on the quotient $\mathbb{Z}_p/\mathbb{Z}^*_p$ and easily obtain the simple tree on the quotient space. Also the measure $\mathrm{pr}_*\big(\tau^{\alpha}_{\mathbb{Z}_p}(x) \otimes \tau^{\beta}_{\mathbb{Z}_p}(x)\big)$ is $\mathbb{Z}^*_p$-invariant. Then dividing by $\mathbb{Z}^*_p$, we get similar tree for $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}^*_p$. Now we further divide the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ by $\mathbb{Z}^*_p \ltimes \mathbb{Z}_p$. This semi-direct product is isomorphic to the subgroup of

$PGL_2(\mathbb{Z}_p)$ whose element are of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Our measure $\tau_p^{\alpha,\beta}$ is not invariant under $\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$, but we can project it down to $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$, we denote the image measure by $\tau_p^{(\alpha)\beta}$. Note that it is not symmetric in the two parameters α and β. Then we have a Markov chain on the tree $\coprod_{n\geq 0} \mathbb{P}^1(\mathbb{Z}/p^n)/(\mathbb{Z}/p^n)^* \ltimes (\mathbb{Z}/p^n)$, it is called the p-adic β-chain.

1.1 Quotients $\mathbb{Z}_p/\mathbb{Z}_p^*$ and $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$

1.1.1 $\mathbb{Z}_p/\mathbb{Z}_p^*$

Every p-adic integer can be written as a power series in p and such a representation is unique. Namely we have

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/(p^n) = \left\{ a_0 + a_1 p + a_2 p^2 + \cdots \,\middle|\, 0 \leq a_j < p \ (j \geq 0) \right\}.$$

We here show that the ring of p-adic integers $\mathbb{Z}_p$ can be identified with the paths in a tree starting from an origin. For example let us consider the case $p = 3$. All element in $\mathbb{Z}_3$ have a series expansion in 3. There are three choices for the constant term a_0, and each one of those has also three choices for a_1, and so on. We obtain a tree in Fig. 1.1 in this way and regard a 3-adic integer as a path in the tree from the origin O.

Note that for an invertible element in $\mathbb{Z}_p$, we have $a_0 \neq 0$. Hence, similarly, the set of all invertible elements $\mathbb{Z}_p^*$ in $\mathbb{Z}_p$ can be expressed as

$$\mathbb{Z}_p^* = \varprojlim \bigl(\mathbb{Z}/(p^n)\bigr)^* = \left\{ a_0 + a_1 p + a_2 p^2 + \cdots \,\middle|\, 0 \leq a_j < p \ (j \geq 0), a_0 \neq 0 \right\}.$$

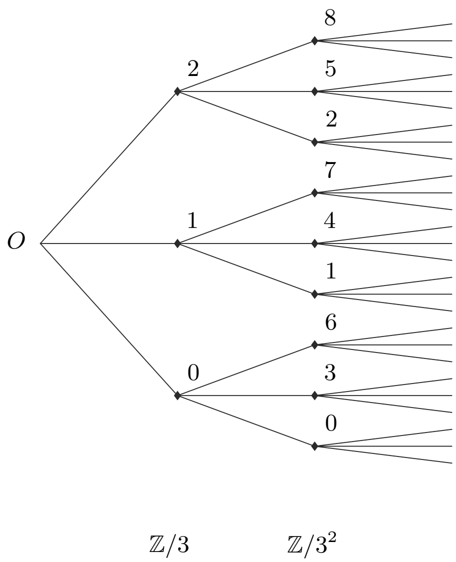


Fig. 1.1. $\mathbb{Z}_3$

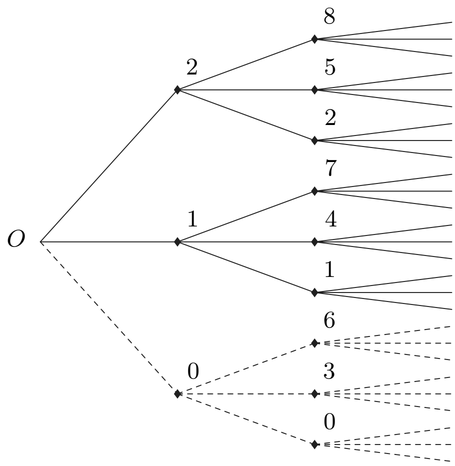


Fig. 1.2. $\mathbb{Z}_3^*$

We remark that the group $\mathbb{Z}_p^*$ is a multiplicative group, while $\mathbb{Z}_p$ is an additive group. It is clear that the tree corresponding to $\mathbb{Z}_p^*$ is obtained by removing a first branch of the tree corresponding to $\mathbb{Z}_p$ because $a_0 \neq 0$ (see Fig. 1.2 in the case $p = 3$).

The group $\mathbb{Z}_p^*$ acts on $\mathbb{Z}_p$ by multiplication. Let us consider the quotient $\mathbb{Z}_p/\mathbb{Z}_p^*$ by the action and give a tree corresponding to $\mathbb{Z}_p/\mathbb{Z}_p^*$. In the tree of $\mathbb{Z}_p$, all branches corresponding to $a_0 \neq 0$ are clearly equivalent to $\mathbb{Z}_p^*$. Hence the tree of $\mathbb{Z}_p/\mathbb{Z}_p^*$ have two branches in the first stage. If $a_0 = 0$, we have also two choices; either $a_1 = 0$ or $a_1 \neq 0$. Then all branches corresponding to $a_1 \neq 0$ are equivalent to $p\mathbb{Z}_p^*$. Continuing this procedure we obtain the tree of $\mathbb{Z}_p/\mathbb{Z}_p^*$ in Fig. 1.3 and have $\mathbb{Z}_p/\mathbb{Z}_p^* = \{0\} \cup \bigsqcup_{n\geq 0} p^n\mathbb{Z}_p^* \simeq \{0\} \cup p^{\mathbb{N}}$. Here we denote by $\mathbb{N}$ the set of all *non-negative integers.*

1.1.2 $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$

Next we investigate the tree of the projective line $\mathbb{P}^1(\mathbb{Q}_p)$, which is defined by the set of all equivalent classes of $\mathbb{Q}_p \times \mathbb{Q}_p \backslash \{(0,0)\}$ under the equivalent relation $\sim$. Here $(x, y) \sim (x', y')$ means that there exists some $c \in \mathbb{Q}_p^*$ such that $x' = cx$ and $y' = cy$. Since $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$, we have clearly $\mathbb{P}^1(\mathbb{Z}_p) = \mathbb{P}^1(\mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$ and write

$$\mathbb{P}^1(\mathbb{Z}_p) = \{0 = (1:0)\} \cup \{(1:x) \,|\, x \in \mathbb{Q}_p^*\} \cup \{\infty = (0:1)\}. \tag{1.1}$$

As is the cases of $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$, $\mathbb{P}^1(\mathbb{Z}_p)$ is also expressed as the inverse limit; $\mathbb{P}^1(\mathbb{Z}_p) = \varprojlim \mathbb{P}^1(\mathbb{Z}/p^n)$. Then we will also obtain the tree corresponding to

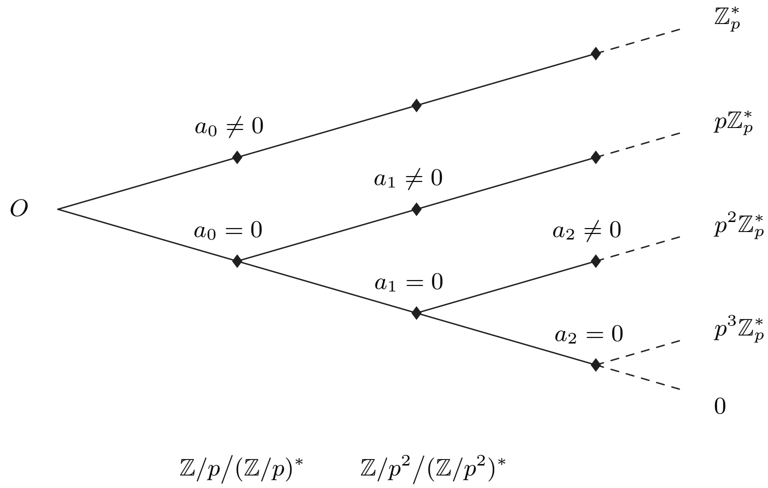


Fig. 1.3. $\mathbb{Z}_p/\mathbb{Z}_p^* \simeq p^{\mathbb{N}} \cup \{0\}$

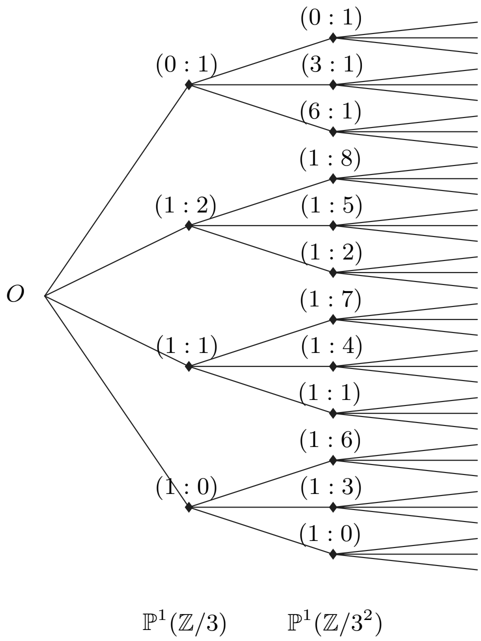


Fig. 1.4. $\mathbb{P}^1(\mathbb{Z}_3)$

$\mathbb{P}^1(\mathbb{Z}_p)$. Since the cardinality of $\mathbb{P}^1(\mathbb{Z}/p)$ is just $p+1$ (that is, $0 = (1:0)$, $(1:1), \ldots, (1:p-1)$ and $\infty = (0:1)$), the tree of $\mathbb{P}^1(\mathbb{Z}_p)$ has $p+1$ branches at the first stage. Then each branch has p sub-branches. Hence we obtain the tree in Fig. 1.4.

The action of the unit group $\mathbb{Z}_p^*$ on the projective line $\mathbb{P}^1(\mathbb{Z}_p)$ is given by

$$\mathbb{P}^1(\mathbb{Z}_p) \times \mathbb{Z}_p^* \ni ((x:y), a) \longmapsto (ax:y) = (x:a^{-1}y) \in \mathbb{P}^1(\mathbb{Z}_p).$$

Similarly, the tree corresponding to the quotient $\mathbb{P}^1(\mathbb{Z}_p)/\mathbb{Z}_p^*$ can be obtained as the case of $\mathbb{Z}_p/\mathbb{Z}_p^*$. Further the group $PGL_2(\mathbb{Z}_p)$ also acts on $\mathbb{P}^1(\mathbb{Z}_p)$ as follows;

$$\mathbb{P}^1(\mathbb{Z}_p) \times PGL_2(\mathbb{Z}_p) \ni ((x:y), \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \longmapsto (ax+cy : bx+dy) \in \mathbb{P}^1(\mathbb{Z}_p).$$

Then it is easy to see that the stabilizer group Stab(0) of $0 = (1:0)$ is given by

$$\mathrm{Stab}(0) = \left\{ \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \in PGL(\mathbb{Z}_p) \,\middle|\, c \in \mathbb{Z}_p,\ d \in \mathbb{Z}_p^* \right\}.$$

This shows that Stab(0) is isomorphic to the semi-direct product $\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ by the isomorphic map

$$\mathrm{Stab}(0) \ni \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \longmapsto (d, c) \in \mathbb{Z}_p^* \ltimes \mathbb{Z}_p$$

and, hence, we obtain a smaller quotient $\mathbb{P}^1(\mathbb{Z}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ of $\mathbb{P}^1(\mathbb{Z}_p)$ than $\mathbb{P}^1(\mathbb{Z}_p)/\mathbb{Z}_p^*$. From the tree of $\mathbb{P}^1(\mathbb{Z}_p)/\mathbb{Z}_p^*$, we obtain the following tree corresponding to $\mathbb{P}^1(\mathbb{Z}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ since two elements $(p^n : 1)$ and $(1:1)$ are equivalent under the action of $\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ for all $n \in \mathbb{N}$ (Figs. 1.5 and 1.6). (In fact, $(p^n:1) \cdot (1, 1-p^n) = (1:1)$ and clearly $(1, 1-p^n) \in \mathbb{Z}_p^* \ltimes \mathbb{Z}_p$.)

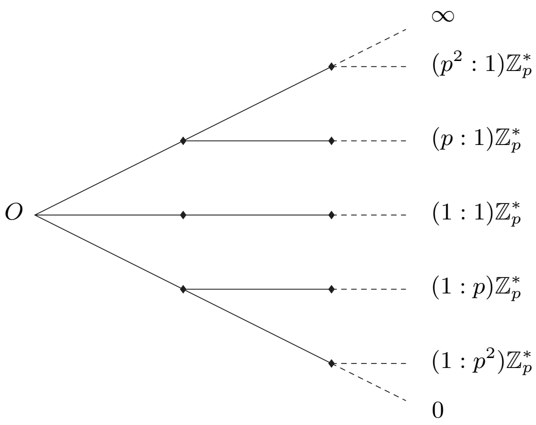


Fig. 1.5. $\mathbb{P}^1(\mathbb{Z}_p)/\mathbb{Z}_p^* \simeq p^{\mathbb{Z}} \cup \{0, \infty\}$

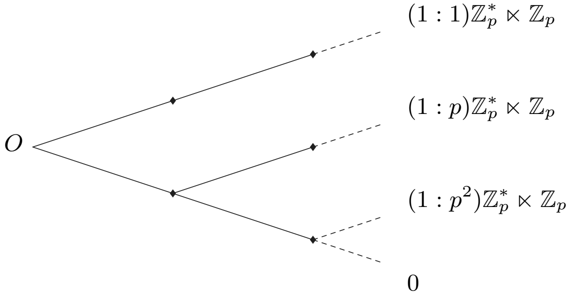


Fig. 1.6. $\mathbb{P}^1(\mathbb{Z}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p \simeq p^{\mathbb{N}} \cup \{0\}$

1.2 γ-Measure on $\mathbb{Q}_p$

1.2.1 p-γ-Integral

Let p be a finite prime. It is known that $\mathbb{Q}_p$ is a locally compact topological additive group. Let dx be the Haar measure on $\mathbb{Q}_p$ normalized by $dx(\mathbb{Z}_p) = 1$. Since dx is invariant under the addition and any open set will be represented as disjoint union of sets such as $q + p^n\mathbb{Z}_p$, we note that $dx(q + p^n\mathbb{Z}_p) = p^{-n}$ for all $n \in \mathbb{Z}$. Let $\phi_{\mathbb{Z}_p}$ be the characteristic function of $\mathbb{Z}_p$;

$$\phi_{\mathbb{Z}_p}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p, \\ 0 & \text{if } x \notin \mathbb{Z}_p. \end{cases}$$

We also let d^*x be the Haar measure on the multiplicative group $\mathbb{Q}_p^*$ normalized by $d^*x(\mathbb{Z}_p^*) = 1$. Since $d(a \cdot x) = |a|_p dx$ for any $a \in \mathbb{Q}_p^*$, the measure $dx/|x|_p$ is also invariant under the multiplication. Since the Haar measure on $\mathbb{Q}_p^*$ is unique up to positive constant multiplication, it holds that $d^*x = c \cdot dx/|x|_p$ for some $c > 0$. Actually,

$$\begin{aligned} c &= \frac{dx}{|x|_p}(\mathbb{Z}_p^*)^{-1} = dx(\mathbb{Z}_p^*)^{-1} \qquad\qquad (|\mathbb{Z}_p^*|_p = 1) \\ &= \big(dx(\mathbb{Z}_p) - dx(p\mathbb{Z}_p)\big)^{-1} \\ &= (1 - p^{-1})^{-1} \\ &= \zeta_p(1), \end{aligned}$$

where $\zeta_p(s) := (1 - p^{-s})^{-1}$ is the local zeta function at p. The function $\zeta_p(s)$ is expressed as the following *p-γ-integral*:

$$\zeta_p(s) = \int_{\mathbb{Q}_p^*} \phi_{\mathbb{Z}_p}(x)|x|_p^s d^*x. \tag{1.2}$$

In fact since $\mathbb{Z}_p = \bigsqcup_{n \geq 0} p^n\mathbb{Z}_p^*$, we have

$$\int_{\mathbb{Q}_p^*} \phi_{\mathbb{Z}_p}(x)|x|_p^s d^*x = \sum_{n \geq 0} \int_{p^n\mathbb{Z}_p^*} |x|_p^s d^*x = \sum_{n \geq 0} p^{-ns} = (1 - p^{-s})^{-1} = \zeta_p(s).$$

1.2.2 η-γ-Integral

What is the real analogue of this, or what is the "characteristic function $\phi_{\mathbb{Z}_\eta}$ of $\mathbb{Z}_\eta$"? From the complete Riemann zeta function, the local zeta function at the real prime $p = \eta$ is defined by $\zeta_\eta(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ where $\Gamma(s)$ is the gamma function. Suppose $\zeta_\eta(s)$ has the similar *η-γ-integral*

$$\zeta_\eta(s) = \int_{\mathbb{Q}_\eta^*} \phi_{\mathbb{Z}_\eta}(x)|x|_\eta^s d^*x, \tag{1.3}$$

where $d^*x = dx/|x|_\eta$ is the invariant measure on $\mathbb{Q}_\eta^* = \mathbb{R}^*$ and $|x|_\eta$ is the usual absolute value. By the Mellin inversion formula, the mysterious function $\phi_{\mathbb{Z}_\eta}$ should be given by

$$\begin{aligned}\phi_{\mathbb{Z}_\eta}(x) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \zeta_\eta(s)|x|^{-s}ds && (\sigma > 0)\\ &= \sum_{n\geq 0}\Big(\operatorname*{Res}_{s=-2n}\zeta_\eta(s)\Big)|x|_\eta^{2n} && (\sigma \to -\infty)\\ &= \sum_{n\geq 0}\frac{(-1)^n}{n!}\pi^n|x|_\eta^{2n}.\end{aligned}$$

Hence we have $\phi_{\mathbb{Z}_\eta}(x) = e^{-\pi x^2}$.

1.2.3 γ-Measure on $\mathbb{Q}_p$

From the γ-integral (1.2) and (1.3), for $\beta > 0$ and all prime $p \geq \eta$ (this means that $p = \eta, 2, 3, 5, \ldots$), we define the *$\gamma$-measure* on $\mathbb{Q}_p$ by

$$\tau_{\mathbb{Z}_p}^\beta := \phi_{\mathbb{Z}_p}(x)|x|_p^\beta \frac{d^*x}{\zeta_p(\beta)}.$$

Then $\tau_{\mathbb{Z}_p}^\beta$ is a probability measure on $\mathbb{Q}_p$; $\int_{\mathbb{Q}_p}\tau_{\mathbb{Z}_p}^\beta = 1$. The name of the γ-measure is given from the gamma function. If $p = \eta$, $\tau_{\mathbb{Z}_\eta}^\beta$ is the Gaussian probability measure, which is invariant under $\mathbb{Z}_\eta^* = \{\pm 1\}$. If $p \neq \eta$, then $\tau_{\mathbb{Z}_p}^\beta$ is also invariant under the action $\mathbb{Z}_p^*$, whence gives a probability measure on $\mathbb{Z}_p/\mathbb{Z}_p^*$. Notice that for all $n \geq 0$, we have $\tau_{\mathbb{Z}_p}^\beta(p^n\mathbb{Z}_p^*) = p^{-n\beta}(1-p^{-\beta})$ (see Fig. 1.7).

1.3 β-Measure on $\mathbb{P}^1(\mathbb{Q}_p)$

1.3.1 The Projective Space $\mathbb{P}^1(\mathbb{Q}_p)$

We next define a measure on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$, which is called β-measure. Let $V(\mathbb{Q}_p) := \mathbb{Q}_p \times \mathbb{Q}_p$ be the plane and $V^*(\mathbb{Q}_p) := \{(x, y) \in V(\mathbb{Q}_p) \mid (x, y) \neq (0, 0)\}$. We define a symplectic form on $V(\mathbb{Q}_p)$ by

$$\langle (x_1, y_1), (x_2, y_2)\rangle := x_1y_2 - y_1x_2 \qquad (x_1, y_1), (x_2, y_2) \in V(\mathbb{Q}_p)$$

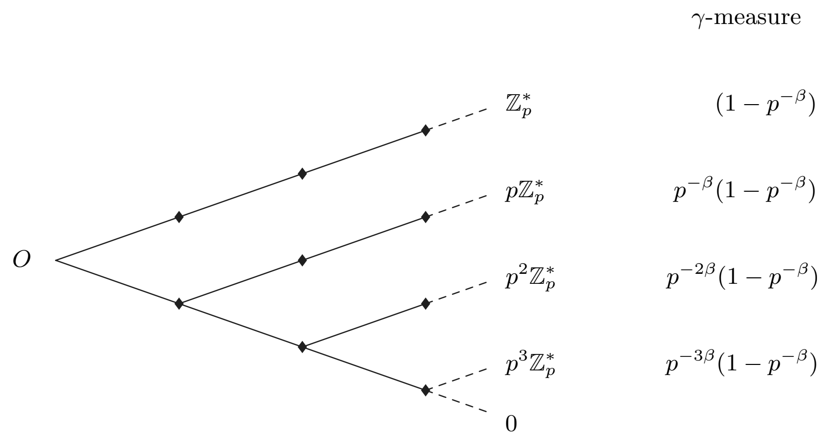


Fig. 1.7. γ-measure on $\mathbb{Z}_p/\mathbb{Z}_p^*$

and an absolute value on $V(\mathbb{Q}_p)$ by

$$|x,y|_p = |(x,y)|_p := \begin{cases} \max\{|x|_p, |y|_p\} & \text{if } p \neq \eta, \\ \sqrt{|x|_\eta^2 + |y|_\eta^2} & \text{if } p = \eta \end{cases} \qquad (x,y) \in V(\mathbb{Q}_p).$$

For $p \neq \eta$, put

$$V(\mathbb{Z}_p) := \mathbb{Z}_p \times \mathbb{Z}_p,$$
$$V^*(\mathbb{Z}_p) := \{(x,y) \in \mathbb{Z}_p \mid |x,y|_p = 1\}.$$

Then the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ is expressed as $\mathbb{P}^1(\mathbb{Q}_p) = V^*(\mathbb{Q}_p)/\mathbb{Q}_p^* = V^*(\mathbb{Z}_p)/\mathbb{Z}_p^*$. For all $p \geq \eta$, there is a canonical distance function $\rho_p : \mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_p) \to [0,1]$ defined by

$$\rho_p\big((x_1 : y_1), (x_2 : y_2)\big) := \frac{|x_1 y_2 - y_1 x_2|_p}{|x_1, y_1|_p \cdot |x_2, y_2|_p}$$

The function ρ_p is well-defined. Namely, the value $\rho_p\big((x_1 : y_1), (x_2 : y_2)\big)$ is determined independently of the choices of (x_1, y_1) and (x_2, y_2) in the equivalence class. The following properties are easily obtained.

$$\begin{aligned}
&(i) \quad \rho_p\big(v_1, v_2\big) \in [0,1], \\
&(ii) \quad \rho_p\big(v_1, v_2\big) = 0 \iff v_1 = v_2, \\
&(iii) \quad \rho_p\big(v_1, v_2\big) = \rho_p\big(v_2, v_1\big), \\
&(iv) \quad \rho_p\big(v_1, v_3\big) \leq \rho_p\big(v_1, v_2\big) + \rho_p\big(v_2, v_3\big)
\end{aligned}$$

Hence ρ_p is a metric on $\mathbb{P}^1(\mathbb{Q}_p)$. Further if $p \neq \eta$, we have

$$\begin{aligned} (iv) \quad & \rho_p\big(v_1, v_3\big) \leq \max\big\{\rho_p\big(v_1, v_2\big), \rho_p\big(v_2, v_3\big)\big\}, \\ (v) \quad & \rho_p(v_1, v_2) \leq p^{-n} \iff v_1 \equiv v_2 \pmod{p^n}. \end{aligned}$$

For $p = \eta$, we have

$$\rho_\eta(v_1, v_2) = |\sin\theta|,$$

where θ is the angle between the lines v_1 and v_2. Let us denote $\rho_\infty(x)$ (resp. $\rho_0(x)$) the distance with respect to ρ_p between $(1 : x) \in \mathbb{P}^1(\mathbb{Q}_p)$ and $\infty = (0 : 1)$ (resp. $0 = (1 : 0)$). Namely,

$$\begin{aligned} \rho_\infty(x) :&= \rho_p\big((0:1), (1:x)\big) = |1, x|_p^{-1}, \\ \rho_0(x) :&= \rho_p\big((1:0), (1:x)\big) = |1, x^{-1}|_p^{-1}. \end{aligned}$$

It is easy to see that for $p \neq \eta$

$$\rho_\infty(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p, \\ |x|_p^{-1} & \text{if } x \in \mathbb{Q}_p \backslash \mathbb{Z}_p, \end{cases} \qquad \rho_0(x) = \begin{cases} |x|_p & \text{if } x \in \mathbb{Z}_p, \\ 1 & \text{if } x \in \mathbb{Q}_p \backslash \mathbb{Z}_p, \end{cases} \tag{1.4}$$

and for $p = \eta$

$$\rho_\infty(x) = (1 + x^2)^{-\frac{1}{2}}, \qquad \rho_0(x) = (1 + x^{-2})^{-\frac{1}{2}}. \tag{1.5}$$

Finally we have

$$\begin{aligned} \max\{\rho_\infty(x), \rho_0(x)\} = 1 \qquad & \text{if } p \neq \eta, \\ \rho_\infty(x)^2 + \rho_0(x)^2 = 1 \qquad & \text{if } p = \eta \end{aligned}$$

and for all $p \geq \eta$

$$|x|_p = \frac{\rho_0(x)}{\rho_\infty(x)}.$$

1.3.2 β-Integral

For $p \geq \eta$, we put

$$\zeta_p(\alpha, \beta) := \frac{\zeta_p(\alpha)\zeta_p(\beta)}{\zeta_p(\alpha + \beta)}.$$

Then, for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, $\zeta_p(\alpha, \beta)$ is expressed as the following *β-integral*;

$$\zeta_p(\alpha, \beta) = \int_{\mathbb{Q}_p^*} \rho_\infty(x)^\alpha \rho_0(x)^\beta d^*x. \tag{1.6}$$

Let us prove this. If $p \neq \eta$, note that for $n \in \mathbb{N}$ we have from (1.4)

$$\rho_\infty(x)^\alpha \rho_0(x)^\beta = \begin{cases} p^{-n\beta} & \text{if } x \in p^n \mathbb{Z}_p^*, \\ 1 & \text{if } x \in \mathbb{Z}_p^*, \\ p^{-n\alpha} & \text{if } x \in p^{-n}\mathbb{Z}_p^*. \end{cases}$$

Hence the right hand side of (1.6) is equal to

$$\sum_{n\geq 1} p^{-n\alpha} + 1 + \sum_{n \geq 1} p^{-n\beta} = \frac{1-p^{-\alpha-\beta}}{(1-p^{-\alpha})(1-p^{-\beta})} = \frac{\zeta_p(\alpha)\zeta_p(\beta)}{\zeta_p(\alpha+\beta)}.$$

On the other hand if $p = \eta$, from (1.5), the right hand side of (1.6) can be written as

$$\begin{aligned} \int_{\mathbb{R}^*} (1+x^2)^{-\frac{\alpha}{2}}(1+x^{-2})^{-\frac{\beta}{2}} d^*x &= \int_0^\infty (1+x)^{-\frac{\alpha}{2}}(1+x^{-1})^{-\frac{\beta}{2}} d^*x \\ &= B\Big(\frac{\alpha}{2}, \frac{\beta}{2}\Big) = \frac{\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(\frac{\alpha+\beta}{2}\big)} = \frac{\zeta_\eta(\alpha)\zeta_\eta(\beta)}{\zeta_\eta(\alpha+\beta)} \end{aligned}$$

where $B(\alpha, \beta)$ is the beta function. Hence we obtain formula (1.6).

1.3.3 β-Measure on $\mathbb{P}^1(\mathbb{Q}_p)$

From the β-integral (1.6), for $\alpha > 0$ and $\beta > 0$, we define the *β-measure* by

$$\tau_p^{\alpha,\beta} := \rho_\infty(x)^\alpha \rho_0(x)^\beta \frac{d^*x}{\zeta_p(\alpha, \beta)}.$$

The name of the β-measure is given from the beta function. Since the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ is expressed as (1.1), $\tau_p^{\alpha,\beta}$ gives a probability measure on $\mathbb{P}^1(\mathbb{Q}_p)$ for all $p \geq \eta$. Further since $\tau_p^{\alpha,\beta}$ is invariant under the action of $\mathbb{Z}_p^*$, it is a probability measure on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$. Here we can take p both finite or real prime. If $p \neq \eta$, we have

$$\tau_p^{\alpha,\beta}((1:p^n)\mathbb{Z}_p^*) = \frac{(1-p^{-\alpha})(1-p^{-\beta})}{(1-p^{-\alpha-\beta})} \times \begin{cases} p^{n\alpha} & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ p^{-n\beta} & \text{if } n > 0. \end{cases}$$

(See Fig. 1.8.) Note that while $\tau_p^{\alpha,\beta}$ is not invariant under the action of $\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$, we can project $\tau_p^{\alpha,\beta}$ to a probability measure on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$. It holds that for $p \neq \eta$

$$\tau_p^{\alpha,\beta}((1:p^n)\mathbb{Z}_p^* \ltimes \mathbb{Z}_p) = \frac{(1-p^{-\alpha})(1-p^{-\beta})}{(1-p^{-\alpha-\beta})} \times \begin{cases} (1-p^{-\alpha})^{-1} & \text{if } n = 0, \\ p^{-n\beta} & \text{if } n > 0 \end{cases}$$

since, for $n = 0$, we have $1 + \sum_{n>0} p^{-n\alpha} = (1-p^{-a})^{-1}$ (see Fig. 1.9).

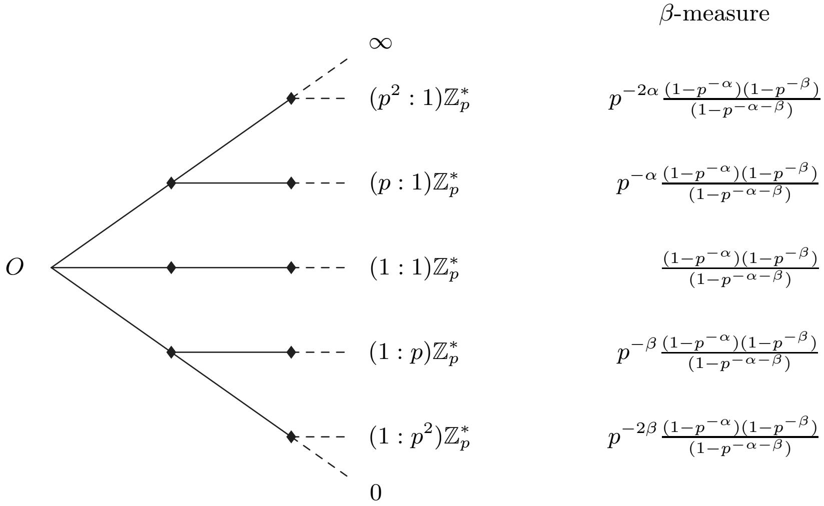


Fig. 1.8. β-measure on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$

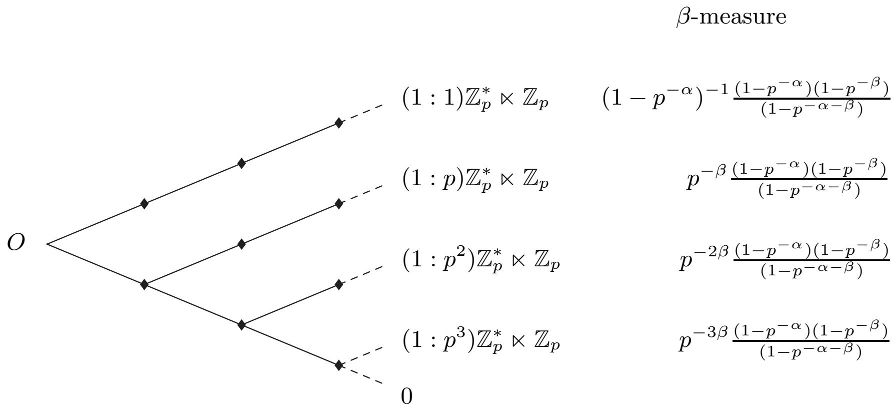


Fig. 1.9. β-measure on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$

1.4 Remarks on the γ and β-Measure

1.4.1 β-Measure Gives γ-Measure

The β-measure is more basic than the γ-measure. Actually the γ-measure is obtained by taking the limit $\alpha \to \infty$ of the β-measure. To see this, we first consider the case of a finite prime $p \neq \eta$. Since $\rho_0(x)^\beta = |x|_p^\beta$ on $\mathbb{Z}_p$ and

$$\rho_\infty(x)^\alpha \to \phi_{\mathbb{Z}_p}(x),$$
$$\zeta_p(\alpha, \beta) \to \zeta_p(\beta)$$

as $\alpha \to \infty$, it is clear that the β-measure converges to the γ-measure. For the real prime $p = \eta$, remark that

$$\tau_\eta^{\alpha,\beta}(x) \to \delta_0(x) \qquad (\alpha \to \infty),$$

where $\delta_0(x)$ is the delta measure supported at 0. Hence we consider the following scaling limit. By the Stirling formula we have

$$\zeta_\eta(\alpha,\beta) = \frac{\zeta_\eta(\alpha)}{\zeta_\eta(\alpha+\beta)}\zeta_\eta(\beta) \sim \left(\frac{\alpha}{2\pi}\right)^{-\frac{\beta}{2}}\zeta_\eta(\beta) \qquad (\alpha \to \infty).$$

This yields that

$$\begin{aligned}\tau_\eta^{\alpha,\beta}\left(\left(\frac{\alpha}{2\pi}\right)^{-\frac{1}{2}} \cdot x\right) &= \left(1+\frac{2\pi}{\alpha}x^2\right)^{-\frac{\alpha}{2}}\left(1+\frac{\alpha}{2\pi}x^{-2}\right)^{-\frac{\beta}{2}}\frac{d^*x}{\zeta_\eta(\alpha,\beta)} \\ &\sim e^{-\pi x^2}\left(\frac{2\pi}{\alpha}+x^{-2}\right)^{-\frac{\beta}{2}}\frac{d^*x}{\zeta_\eta(\beta)} \\ &\to e^{-\pi x^2}|x|_\eta^\beta\frac{d^*x}{\zeta_\eta(\beta)}\end{aligned}$$

as $\alpha \to \infty$. Hence we have $\tau_\eta^{\alpha,\beta}\left(\left(\frac{\alpha}{2\pi}\right)^{-\frac{1}{2}} \cdot x\right) \to \tau_{\mathbb{Z}_\eta}^\beta(x)$ as $\alpha \to \infty$. Note that, more generally, we have for $c > 0$

$$\tau_\eta^{\alpha,\beta}\left(\left(\frac{\alpha}{c}\right)^{-\lambda} \cdot x\right) \to \begin{cases}\delta_0(x) & \text{if } \lambda < \dfrac{1}{2}, \\ \left(\dfrac{c}{2\pi}\right)^{\frac{\beta}{2}} e^{-\frac{c}{2}x^2}|x|_\eta^\beta\dfrac{d^*x}{\zeta_\eta(\beta)} & \text{if } \lambda = \dfrac{1}{2}, \\ \delta_\infty(x) & \text{if } \lambda > \dfrac{1}{2},\end{cases} \qquad (\alpha \to \infty).$$

1.4.2 γ-Measure Gives β-Measure

Next we show that the γ-measure gives the β-measure. Let us denote by pr_* the push forward from the set of measures on $V^*(\mathbb{Q}_p)$ onto the set of measures on $\mathbb{P}^1(\mathbb{Q}_p)$. Then the measure $\mathrm{pr}_*(\tau_{\mathbb{Z}_p}^\alpha \otimes \tau_{\mathbb{Z}_p}^\beta)$ gives the β-measure. Actually it holds that

$$\begin{aligned}\int_{(1:x)} \tau_{\mathbb{Z}_p}^\alpha \otimes \tau_{\mathbb{Z}_p}^\beta &= \int_{\mathbb{Q}_p^*} d^*a \cdot \phi_{\mathbb{Z}_p}(a)\frac{|a|_p^\alpha}{\zeta_p(\alpha)}\phi_{\mathbb{Z}_p}(ax)\frac{|ax|_p^\beta}{\zeta_p(\beta)} \\ &= \frac{|x|_p^\beta}{\zeta_p(\alpha)\zeta_p(\beta)}\int_{\mathbb{Q}_p^*} d^*a \cdot \phi_{\mathbb{Z}_p}(a \cdot |1,x|_p)|a|_p^{\alpha+\beta} \\ &= \frac{|x|_p^\beta |1,x|_p^{-\alpha-\beta}}{\zeta_p(\alpha)\zeta_p(\beta)}\zeta_p(\alpha+\beta) \\ &= \rho_\infty(x)^\alpha \rho_0(x)^\beta \frac{1}{\zeta_p(\alpha,\beta)}.\end{aligned}$$

Hence we have $\mathrm{pr}_*(\tau_{\mathbb{Z}_p}^{\alpha} \otimes \tau_{\mathbb{Z}_p}^{\beta}) = \tau_p^{\alpha,\beta}$. Notice that in the second equality we use the relation

$$\phi_{\mathbb{Z}_p}(a)\phi_{\mathbb{Z}_p}(b) = \phi_{\mathbb{Z}_p}(|a,b|_p)$$

where $|a,b|_p$ on the right hand side denote any element of $\mathbb{Q}_p$ of absolute value $|a,b|_p$; this is shown as follows. We have for $p \neq \eta$

$$\phi_{\mathbb{Z}_p}(|a,b|_p) = \begin{cases} 1 & \text{if } a, b \in \mathbb{Z}_p, \\ 0 & \text{otherwise} \end{cases} = \phi_{\mathbb{Z}_p}(a)\phi_{\mathbb{Z}_p}(b)$$

and for $p = \eta$

$$\phi_{\mathbb{Z}_\eta}(|a,b|_\eta) = e^{-\pi(\sqrt{a^2+b^2})^2} = e^{-\pi a^2}e^{-\pi b^2} = \phi_{\mathbb{Z}_\eta}(a)\phi_{\mathbb{Z}_\eta}(b).$$

1.4.3 Special Case $\alpha = \beta = 1$

As a final remark we consider the β-measure at a special value of the parameters, that is, $\alpha = \beta = 1$. For a finite prime $p \neq \eta$, $\tau_p^{1,1}$ gives the unique $PGL_2(\mathbb{Z}_p)$-invariant probability measure on $\mathbb{P}^1(\mathbb{Q}_p)$. For the real prime $p = \eta$, $\tau_\eta^{1,1}$ gives the unique $O(2)$-invariant probability measure on $\mathbb{P}^1(\mathbb{R})$. Note that $O(2) = GL_2(\mathbb{Z}_\eta)$. Similarly, the γ-measure $\tau_{\mathbb{Z}_p}^{\beta}$ at $\beta = 1$ gives the additive measure

$$\tau_{\mathbb{Z}_p}^{1}(x) = \phi_{\mathbb{Z}_p}(x)dx.$$

2

Markov Chains

Summary. In Sect. 2.1 we show there is a bijection between probability measures τ on the boundary space ∂X of a tree X, and Markov chain on X. For each point x on the tree, we consider the set of all the paths going through x and call it the interval $I(x)$. The interval splits into intervals $I(x')$ corresponding to each arrow $x \mapsto x'$, and we give this arrow the probability $\tau(I(x'))/\tau(I(x))$. The sum of the probability is equal to 1. This is a Markov chain. We then give a brief description in Sect. 2.2 of the boundary theory of general transient Markov chains. Let $X = \bigsqcup_n X_n$, $X_0 = \{x_0\}$ be the state space, $P : \bigsqcup_n X_n \times X_{n+1} \to [0,1]$ the transition probability. Then we have

$$\begin{array}{ccc} \text{Probability measure} & \tau_n(x) = (P^*)^n \delta_{x_0}(x) & (x \in X_n), \\ \text{Green kernel} & G(x,y) = P^{m-n}(x,y) & (x \in X_n, y \in X_m), \\ \text{Martin kernel} & K(x,y) = \dfrac{G(x,y)}{G(x_0,y)}. & \end{array}$$

The Martin kernel gives a metric. The sequence $\{y_n\}$ is a Cauchy sequence if $\{K(x,y_n)\}$ is a Cauchy sequence of $\mathbb{R}$ for all x and $\{y_n\} \sim \{y'_n\}$ if $\{K(x,y_n)\} \sim \{K(x,y'_n)\}$. Then we obtain the compactification

$$\overline{X} = \{\text{Cauchy sequence of } X\}/\sim \; = X \sqcup \partial X.$$

Recall the theorem that every super-harmonic function f is equal to K_μ for some μ which is a probability measure on $X \sqcup \partial X$. Here a function f is called super-harmonic if $Pf \geq f$. If $Pf = f$, we call f a harmonic function and μ is a measure supported only on the boundary ∂X. The set $\mathrm{Harm}(X)$ of all harmonic functions on X is divided as

$$\mathrm{Harm}(X) = \mathrm{Harm}(X)_{\text{ext}} \sqcup \mathrm{Harm}(X)_{\text{non-ext}}$$

and the boundary ∂X also decomposes as

$$\partial X = \partial X_{\text{ext}} \sqcup \partial X_{\text{non-ext}}.$$

Here a point $y \in \partial X$ is called extream if $K_{\delta_y} = K(x,y)$ is extream harmonic function. Then there is one-to-one correspondence between the probability measures on $\partial X_{\rm ext}$ and the harmonic functions on X.

2.1 Markov Chain on Trees

2.1.1 Probability Measures on ∂X

Let X be a tree and $x_0 \in X$ the root. For $n \geq 0$, we denote by X_n the set

$$X_n := \{x \in X \,|\, d(x_0, x) = n\}.$$

Then X decomposes as the disjoint union of X_n; $X = \bigsqcup_{n \geq 0} X_n$. Note that $X_0 = \{x_0\}$ and X_n is a finite set. The boundary ∂X of X is defined by the inverse limit of sets X_n or as the collection of all paths starting from the root x_0,

$$\partial X := \varprojlim X_n = \big\{ \tilde{x} = \{x_n\} \,\big|\, x_n \in X_n,\ d(x_n, x_{n+1}) = 1 \big\}.$$

For $x \in X_n$, we denote $I(x) \subset \partial X$, which is called the “interval” of x, by

$$I(x) := \big\{ \tilde{x} = \{x_n\} \in \partial X \,|\, x_n = x \big\}$$

and give a topology in ∂X by regarding the family $\{I(x) \,|\, x \in X\}$ as open base of ∂X.

Let τ be a probability measure on the boundary ∂X. Then we obtain a function $\tau : X \to [0,1]$ defined by $\tau(x) := \tau(I(x))$ and it satisfies

$$\tau(x_0) = 1, \quad \tau(x) = \sum_{\substack{x' \in X_{n+1} \\ x \mapsto x'}} \tau(x') \quad (x \in X_n) \tag{2.1}$$

since $I(x_0) = \partial X$ and $I(x) = \bigsqcup_{x \mapsto x'} I(x')$. Here we write $x \mapsto x'$ instead of $d(x, x') = 1$. Conversely, let τ be a function on the tree X satisfying the condition (2.1). Let $\tau(I(x)) := \tau(x)$. Then τ gives a probability measure on ∂X since each open set of ∂X is expressed as the disjoint union of some intervals $I(x)$. Therefore we have the following one-to-one correspondence;

$$\mathfrak{M}_1(\partial X) \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{the function on } X \text{ satisfying} \\ \text{the condition (2.1)} \end{array} \right\}.$$

Here $\mathfrak{M}_1(Y)$ denotes the set of all probability measure on Y.

Now given such a τ, we define the probability of going from x to x' by $P(x \mapsto x') := \tau(x')/\tau(x)$. It is clear from (2.1) that

$$\sum_{\substack{x' \in X \\ x \mapsto x'}} P(x \mapsto x') = 1 \qquad (x \in X). \tag{2.2}$$

Hence we have a Markov chain (the condition (2.2) is called the Markov condition). Namely, we have a tree X, which is called the "state space", and the function

$$P : \bigsqcup_{n \geq 0} X_n \times X_{n+1} \longrightarrow [0,1]$$

satisfying the condition (2.2). We call such a function P the "transition probability". Conversely, if we are given a tree X and a function P satisfying the Markov condition, we can get a probability measure on ∂X as follows; For any $x \in X$, we have the unique path $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x$ from x_0 to x. Define the function $\tau : X \to [0,1]$ by

$$\tau(x) := P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x_n = x).$$

Then we have from (2.2) that

$$\begin{aligned} \sum_{x \mapsto x'} \tau(x') &= \sum_{x \mapsto x'} P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) P(x \mapsto x') \\ &= P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) \sum_{x \mapsto x'} P(x \mapsto x') \\ &= \tau(x). \end{aligned}$$

Hence the function $\tau(x)$ satisfies the condition (2.1) and $\tau(I(x)) := \tau(x)$ gives a probability measure on ∂X. We call τ the harmonic measure of P. Hence we obtain the following one-to-one correspondence;

$$\mathfrak{M}_1(\partial X) \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Markov chain on } X; \\ \text{transition probability } P \end{array} \right\}.$$

2.1.2 Hilbert Spaces

Let P be a transition probability and τ its harmonic measure on ∂X. Then we can obtain the probability measure τ_n on X_n by

$$\tau_n(x) = \tau(I(x)) := P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) \qquad (x \in X_n),$$

where $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x$ is the unique path from x_0 to x. This can be also written as $\tau_n(x) = (P^*)^n \delta_{x_0}(x)$ where P^* is the adjoint of P and δ_{x_0} is the delta function at x_0 (see the next section). Hence, for all $n \geq 0$, we obtain the Hilbert space

$$H_n := \ell^2(X_n, \tau_n) = \big\{ f : X_n \to \mathbb{C} \,\big|\, ||f||_{H_n} < \infty \big\},$$

where $||f||_{H_n} := (f,f)_{H_n}^{1/2}$ and $(\cdot,\cdot)_{H_n}$ is the inner product of H_n defined by

$$(f,g)_{H_n} := \sum_{x \in X_n} f(x) \overline{g(x)} \tau_n(x).$$

For each $n \geq 0$, we have an embedding $H_n \hookrightarrow H_{n+1}$ defined by

$$H_n \ni \varphi \longmapsto \varphi' \in H_{n+1}; \qquad \varphi'(x') := \varphi(x),$$

where $x \in X_n$ is the unique element such that $x \mapsto x'$. This is an unitary embedding, that is, it preserves the inner product, and we hence identify H_n with a subspace of H_{n+1}. On the other hand we have the orthogonal projection from H_{n+1} onto the subspace H_n

$$H_{n+1} \ni \varphi' \longmapsto \varphi = P\varphi' \in H_n; \qquad P\varphi'(x) := \sum_{\substack{x' \in X_{n+1} \\ x \mapsto x'}} P(x \mapsto x')\varphi'(x').$$

In fact, we can easily show that $\varphi' - P\varphi' \in {H_n}^{\perp} := \{f \in H_{n+1} \,|\, (f,g)_{H_{n+1}} = 0$ for all $g \in H_n\}$.

Since we have a probability measure τ on ∂X, we have another Hilbert space

$$H := \ell^2(\partial X, \tau) = \big\{ f : \partial X \to \mathbb{C} \,\big|\, ||f||_H < \infty \big\},$$

where $||f||_H := (f,f)_H^{1/2}$ and $(\cdot,\cdot)_H$ is the inner product of H defined by

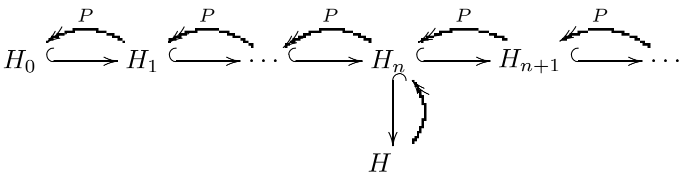
$$(f,g)_H := \int_{\partial X} f(\tilde{x})\overline{g(\tilde{x})}\tau(d\tilde{x}).$$

There is also an unitary embedding map $H_n \hookrightarrow H$ for all $n \geq 0$ defined by

$$H_n \ni \varphi \longmapsto \tilde{\varphi} \in H; \qquad \tilde{\varphi}(\tilde{x}) := \varphi(x_n)$$

with $\tilde{x} = \{x_n\}$ and this is an unitary embedding. The orthogonal projection from H onto H_n is given as follows;

$$H \ni \tilde{\varphi} \longmapsto \varphi \in H_n; \qquad \varphi(x_n) := \frac{1}{\tau_n(x_n)} \int_{I(x_n)} \tilde{\varphi}(\tilde{x})\tau(d\tilde{x}).$$



2.1.3 Symmetric *p*-Adic *β*-Chain

Let us describe the Markov chains associated to the p-adic trees and measures on them. We first give the symmetric β-chain on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ with β-measure. The set of all points on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ is identified with $X = \mathbb{N} \times \mathbb{N}$, the state space. In fact, let

$$X_n = \big\{ (i,j) \in \mathbb{N} \times \mathbb{N} \,\big|\, \max\{i,j\} = n \big\}.$$

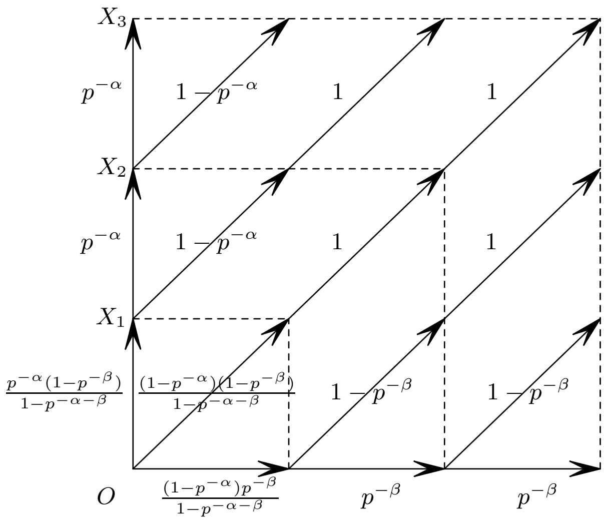


Fig. 2.1. Symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$

Then X_n can be identified with $\mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^*$ by the following correspondence;

$$X_n \ni (i,j) \longmapsto (p^{n-i} : p^{n-j}) \in \mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^*.$$

One can easily obtain the probability measure of each arrow (see Fig. 2.1).

Remember the projection from $\mathbb{P}^1(\mathbb{Q}_p)$ onto $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$. If we want to know the probability measure of an arrow in the tree of $\mathbb{P}^1(\mathbb{Q}_p)$, we divide the probability of the projected arrow in $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ by the number of the arrow of $\mathbb{P}^1(\mathbb{Q}_p)$ corresponding to the given arrow in $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$. For example if $\alpha = \beta = 1$, it it easy to see that the probability of each arrow is given as in Fig. 2.2 (for the case $p = 3$). Note that if $\alpha = \beta = 1$, the β-measure $\tau_p^{1,1}$ is the unique $PGL_2(\mathbb{Z}_p)$-invariant measure. In this case we call this the "random walk". Random means that the probability of each arrow is alway the same at any stage. But this is only $\alpha = \beta = 1$.

2.1.4 Non-Symmetric p-Adic β-Chain

The symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ is still too complicated for us. We next consider the chain on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$. Since this is not symmetric, we call this non-symmetric β-chain. Note that the tree of $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ is obtained by collapsing all of the paths corresponding to $(p^n : 1)\mathbb{Z}_p^*$ for $n \geq 0$ of $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ together. Let

$$X_n = \big\{(i,j) \in \mathbb{N} \times \mathbb{N} \,\big|\, i + j = n\big\}.$$

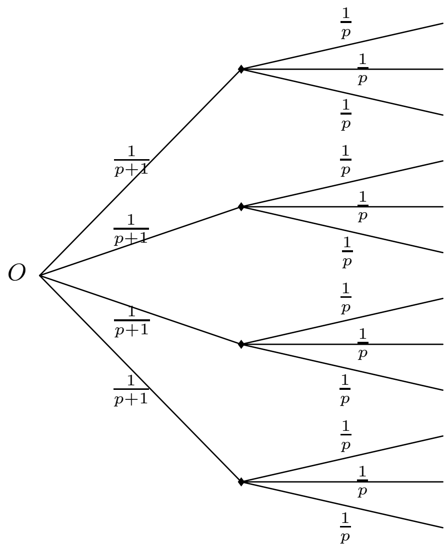


Fig. 2.2. Random walk on $\mathbb{P}^1(\mathbb{Q}_p)$

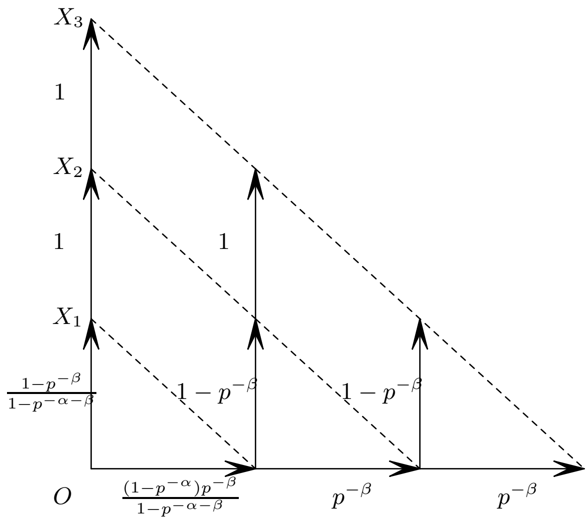


Fig. 2.3. Non-symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$

We also regard $X = \mathbb{N} \times \mathbb{N}$ as the state space by the following correspondence;

$$X_n \ni (i,j) \longmapsto (1 : p^{n-j}) = (1 : p^i) \in \mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^* \ltimes (\mathbb{Z}/p^n).$$

The probability measure is also given in Fig. 2.3. We will concentrate on this chain because it is very simple and will expect a real analogue of the chain.

Notice that the dimension of the Hilbert space $H_n = \ell^2(X_n, \tau_n)$ is given by $\dim H_n = \# X_n = n + 1$. Now H_n is embedding into H_{n+1} and the dimension grows by 1 at each stage. Therefore we conclude that there is a unique function $\varphi_n \neq 0$, up to constant multiplied in $H_n \cap (H_{n-1})^{\perp}$ and obtain the orthogonal decomposition $H_n = \mathbb{C}\varphi_n \oplus H_{n-1}$. Let us decide this function. First it is easy to see that

$$\varphi_1 = \mathbf{1}, \tag{2.3}$$

where $\mathbf{1}$ is the constant function. Next φ_1 is the function on $X_1 = \{(1,0),(0,1)\}$ and satisfies $(\varphi_1, \varphi_0)_{H_1} = 0$. Namely,

$$\varphi_1(1,0)\tau_1(1,0) + \varphi_1(0,1)\tau_1(0,1) = 0.$$

Since $\tau_1(1,0) = (1-p^{-\alpha})p^{-\beta}/(1-p^{-\alpha-\beta})$ and $\tau_1(0,1) = (1-p^{-\beta})/(1-p^{-\alpha-\beta})$, we conclude that

$$\varphi_1(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta} & \text{if } (i,j) = (1,0), \\ -(1-p^{-\alpha}) & \text{if } (i,j) = (0,1). \end{cases} \tag{2.4}$$

Similar on the n-th set $X_n = \{(n,0),(n-1,1),\ldots,(0,n)\}$ for $n \geq 2$, the function φ_n is given by

$$\varphi_n(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta n} & \text{if } (i,j) = (n,0), \\ -p^{\beta(n-1)} & \text{if } (i,j) = (n-1,0), \\ 0 & \text{if } 0 \leq i < n-1. \end{cases} \tag{2.5}$$

By the embedding $H_n \hookrightarrow H_{n+1}$, the function φ_n, which is an element of H_n, can be viewed also as the function on the following spaces H_N for $N > n$. Hence we also obtain the orthogonal decomposition of the N-th layer H_N from (2.3), (2.4) and (2.5);

$$H_N = \bigoplus_{0 \leq m \leq N} \mathbb{C}\varphi_{N,m},$$

where

$$\varphi_{N,0} = \mathbf{1},$$

$$\varphi_{N,1}(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta} & \text{if } 0 < i \leq N, \\ -(1-p^{-\alpha}) & \text{if } i = 0, \end{cases}$$

$$\varphi_{N,m}(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta m} & \text{if } m-1 < i \leq N, \\ -p^{\beta(m-1)} & \text{if } i = m-1, \\ 0 & \text{if } 0 \leq i < m-1, \end{cases} \qquad (m \geq 2).$$

Remember that the function φ_n can be naturally viewed as the element of the boundary space $H = \ell^2(\partial X, \tau)$. Therefore the Hilbert space H is also written as the orthogonal direct sum over all $m \geq 0$;

$$H = \bigoplus_{m \geq 0} \mathbb{C}\varphi_m.$$

Identifying the boundary $\partial X = \mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$, we have

$$\begin{aligned} \varphi_0 &= \mathbf{1}, \\ \varphi_1 &= (1 + p^{\beta} - p^{-\alpha})\phi_{p\mathbb{Z}_p} - (1 - p^{-\alpha})\mathbf{1}, \\ \varphi_m &= p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}\phi_{p^{m-1}\mathbb{Z}_p} \qquad (m \geq 2) \end{aligned}$$

since, say for $m \geq 2$, $\varphi_m = (1 - p^{-\beta})p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}(\phi_{p^{m-1}\mathbb{Z}_p} - \phi_{p^m\mathbb{Z}_p})$.

We will denote in future H_N by $H_{p(N)}^{(\alpha)\beta}$. (The reason why we denote $(\alpha)\beta$ but not α, β is that it is not symmetric for α and β.) The boundary space H is also written as $H_p^{(\alpha)\beta}$. Further we denote the basis $\varphi_{N,m}$ of H_N by $\varphi_{p(N),m}^{(\alpha)\beta}$ and the basis φ_m of H by $\varphi_{p,m}^{(\alpha)\beta}$. We call $\varphi_{p(N),m}^{(\alpha)\beta}$ the p-Hahn basis (an analogue of the Hahn polynomial) and $\varphi_{p,m}^{(\alpha)\beta}$ the p-Jacobi basis (an analogue of the Jacobi polynomial).

2.1.5 p-Adic γ-Chain

Let us consider the γ-measure. Take $\alpha \to \infty$ in either the symmetric β-chain or non-symmetry β-chain. We get the following tree in Fig. 2.4, called the p-adic γ-chain.

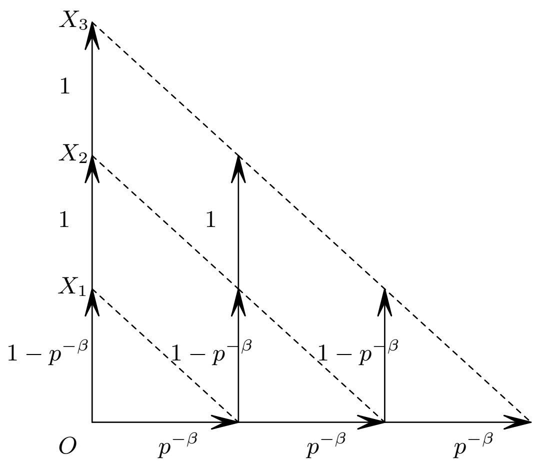


Fig. 2.4. γ-chain on $\mathbb{Z}_p/\mathbb{Z}_p^*$

Similarly we obtain the orthogonal decomposition of H_N and H;

$$H_N := H^{\beta}_{p(N)} = \bigoplus_{0\le m\le N} \mathbb{C}\varphi^{\beta}_{p(N),m},$$
$$H := H^{\beta}_{\mathbb{Z}_p} = \bigoplus_{m\ge 0} \mathbb{C}\varphi^{\beta}_{\mathbb{Z}_p,m},$$

where $\varphi^{\beta}_{p(N),m}$ (resp. $\varphi^{\beta}_{\mathbb{Z}_p,m}$) is the basis of H_N (resp. H) defined by

$$\varphi^{\beta}_{p(N),0} = \mathbf{1},$$
$$\varphi^{\beta}_{p(N),1}(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta} & \text{if } 0 < i \le N, \\ -1 & \text{if } i = 0, \end{cases}$$
$$\varphi^{\beta}_{p(N),m}(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta m} & \text{if } m-1 < i \le N, \\ -p^{\beta(m-1)} & \text{if } i = m-1, \\ 0 & \text{if } 0 \le i < m-1, \end{cases} \qquad (m \ge 2).$$

and

$$\varphi^{\beta}_{\mathbb{Z}_p,0} = \phi_{\mathbb{Z}_p},$$
$$\varphi^{\beta}_{\mathbb{Z}_p,m} = p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}\phi_{p^{m-1}\mathbb{Z}_p} \qquad (m \ge 1).$$

We call $\varphi^{\beta}_{\mathbb{Z}_p,m}$ the p-Laguerre basis, it is the analogue of the Laguerre polynomial.

Note that if $\beta = 1$, the γ-measure can be written as $\tau^1_{\mathbb{Z}_p} = \phi_{\mathbb{Z}_p}(x)|x|^1_p d^*x/\zeta_p(1) = dx$, where dx is the Haar measure of the additive group $\mathbb{Q}_p$ normalized to be a probability measure by $dx(\mathbb{Z}_p) = 1$. This show that $\tau^1_{\mathbb{Z}_p}$ is an "additive" measure. Hence the probability of each arrow in the tree of $\mathbb{Z}_p$ (which is over that of $\mathbb{Z}_p/\mathbb{Z}_p^*$) is given by $1/p$, therefore it is also random walk see Fig. 2.5 (for $p = 3$).

Notice also that if we take the limit $\beta \to \infty$, the γ-measure $\tau^{\beta}_{\mathbb{Z}_p}$ becomes the probability measure on $\mathbb{Z}_p^*$ since $\tau^{\beta}_{\mathbb{Z}_p}(x) \to 0$ for $x \in p\mathbb{Z}_p$. Further if $x \in \mathbb{Z}_p^*$, we have $\tau^{\beta}_{\mathbb{Z}_p}(x) = \phi_{\mathbb{Z}_p^*}(x)d^*x/\zeta_p(\beta) \to d^*x$ and this gives the "multiplicative" measure.

2.2 Markov Chain on Non-Trees

2.2.1 Non-Tree

Now let us consider the real analogue. We already obtain the real analogue of the measure on the boundary, the real analogue of the γ-measure and β-measure. Then what is the real analogue of the Markov chain? We usually

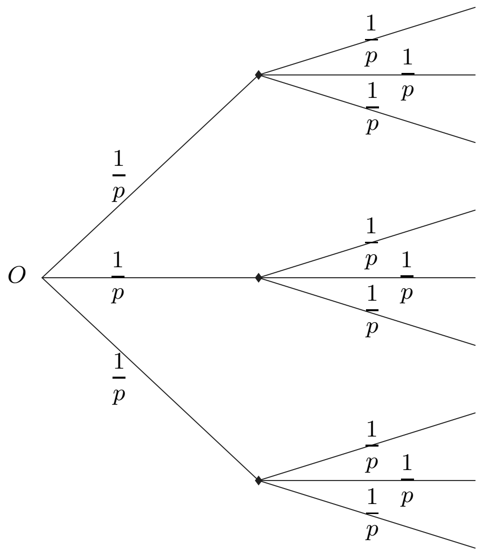


Fig. 2.5. Random walk on $\mathbb{Z}_p$

represent a real number as a path in a "tree". For example, in decimal expansion, each real number is identified with a path from the origin in the $10+1$ regular tree and we obtain $\mathbb{R}$, the set of all real numbers, as the boundary of the tree. Here we sometimes identify two paths, for instance, $1.0000\ldots$ is identified with $0.9999\ldots$. This shows that the boundary is not totally disconnected, hence this is a non-tree (for any tree, the boundary is always totally disconnected). In this section we study the Markov chain on non-trees, which can have continuous boundary.

2.2.2 Harmonic Functions

Let $X = \bigsqcup_{n \geq 0} X_n$, $X_0 = \{x_0\}$ and X_n be a finite set for all $n \geq 0$. We call X the state space. Let $P : \bigsqcup_{n \geq 0} X_n \times X_{n+1} \to [0,1]$ be a transition probability, that is, P satisfies

$$\sum_{x' \in X_{n+1}} P(x, x') = 1 \qquad (x \in X_n). \tag{2.6}$$

Then we says that we have a Markov chain. If for any $x \in X_n$ there exists a sequence $x_0, x_1, \ldots, x_n = x$ such that $x_j \in X_j$ and $P(x_j, x_{j+1}) > 0$, we say that x is reachable from x_0. We assume that every state $x \in X$ is reachable from x_0. The function P can be extended as a function on $X \times X$ by giving 0 if two points x, x' are not connected. Therefore we can regard P as a matrix over $X \times X$.

We also regard P as an operator which acts on $\ell^\infty(X)$, the space of all bounded function on X, as follows;

$$Pf(x) := \sum_{x' \in X} P(x,x')f(x')$$

It is easy to see

$$\begin{aligned} (i) &\quad f \geq 0 \Longrightarrow Pf \geq 0, \\ (ii) &\quad P\mathbf{1} = \mathbf{1} \end{aligned}$$

from the Markov property (2.6).

We have the adjoint operator P^*, which acts on $\ell^1(X)$, defined by

$$P^*\mu(x') := \sum_{x \in X} \mu(x)P(x,x').$$

This operator satisfies

$$\begin{aligned} (i) &\quad \mu \geq 0 \Longrightarrow P^*\mu \geq 0, \\ (ii) &\quad \int_X P^*\mu = \int_X \mu = \sum_{x \in X} \mu(x). \end{aligned}$$

The Laplacian Δ is given by the operator

$$\Delta := \mathbf{1} - P.$$

The function $f : X \to [0,\infty)$ is called harmonic if

$$\Delta f \equiv 0, \qquad f(x_0) = 1.$$

(Here the second condition is a normalization.) Note that the constant function $\mathbf{1}$ is clearly harmonic. Up to a constant multiplication, this is equivalent to the equation

$$f(x) = \sum_{x'} P(x,x')f(x')$$

We denote by $\mathrm{Harm}(X)$ the collection of all harmonic functions. Notice that $\mathrm{Harm}(X)$ is convex. Namely,

$$\begin{matrix} f_0, f_1 \in \mathrm{Harm}(X) \\ \lambda_0, \lambda_1 \geq 0,\ \lambda_0 + \lambda_1 = 1 \end{matrix} \Longrightarrow \lambda_0 f_0 + \lambda_1 f_1 \in \mathrm{Harm}(X).$$

The set $\mathrm{Harm}(X)$ is also compact for the topology of pointwise convergence. If we can take $\lambda_0, \lambda_1 > 0$, then such a function is called non-extremal and we let $\mathrm{Harm}(X)_{\text{non-ext}}$ be the set of all non-extremal harmonic function;

$$\mathrm{Harm}(X)_{\text{non-ext}} := \left\{ \lambda_0 f_0 + \lambda_1 f_1 \,\middle|\, f_0, f_1 \in \mathrm{Harm}(X),\ \lambda_0, \lambda_1 > 0, \lambda_0 + \lambda_1 = 1 \right\}.$$

The harmonic function is called extream if it is not non-extream and we denote by $\mathrm{Harm}(X)_{\text{ext}}$ the set of all extream harmonic functions. Then we obtain

$$\mathrm{Harm}(X) = \mathrm{Harm}(X)_{\text{non-ext}} \sqcup \mathrm{Harm}(X)_{\text{ext}}.$$

This is a basic decomposition of a convex set (see Fig. 2.6).

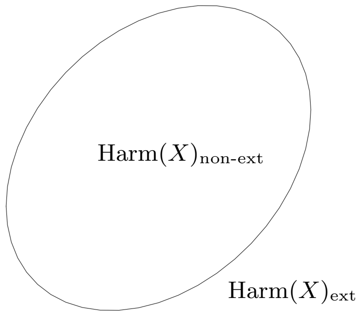


Fig. 2.6. Harm(X)

2.2.3 Martin Kernel

The Green kernel G is given by the operator

$$G := \Delta^{-1} = \sum_{m \geq 0} P^m.$$

If we view P as a matrix on $X \times X$, G can be expressed as follows; Since $P^m(x,y)$ is 0 unless $x \in X_n$ and $y \in X_{n+m}$ for some $n \in \mathbb{N}$, we have

$$G(x,y) = \sum_{x, x_1, \dots, x_m = y} P(x, x_1) \cdots P(x_{m-1}, y)$$

where the sum is over all paths from x to y. Fix a point $y \in X$. Then the function $G(\cdot, y) : X \to [0, \infty)$ has finite support and is essentially harmonic except for the point $x = y$. Namely,

$$G(x,y) = \sum_{x \mapsto x'} P(x, x')G(x', y) \qquad (x \neq y).$$

If $x = y$, we have $G(y,y) = 1$ by the definition. Therefore we conclude that

$$\Delta G(\cdot, y) = \delta_{y,\cdot}.$$

We next define the Martin Kernel K by

$$K(x,y) := \frac{G(x,y)}{G(x_0,y)}.$$

Hence this function will also be harmonic outside of $x = y$ if we regard $K(x,y)$ as a function of x for a fixed $y \in X$. Note that

$$G(x_0, y) \geq G(x_0, x)G(x,y).$$

and we obtain the bound of the Martin Kernel;

$$K(x,y) \le \frac{1}{G(x_0,x)}.$$

Now the Martin metric $d : X \times X \to [0,1]$ is defined by

$$d(y_1,y_2) := \sum_{n\ge 0} \frac{1}{2^{n+1}} \frac{1}{\# X_n} \sum_{x\in X_n} G(x_0,x)|K(x,y_1) - K(x,y_2)|.$$

The sequence $\{x_n\}$ is a Cauchy sequence with respect to the Martin metric if, for every $x \in X$, $\{K(x,x_n)\} \subset \mathbb{R}$ is a Cauchy sequence. We say that two such sequences $\{x_n\}$ and $\{x'_n\}$ are equivalent (we write simply $\{x_n\} \sim \{x'_n\}$) if $d(x_n,x'_n) \to 0$ as $n \to \infty$. This is equivalent to $\{K(x,x_n)\} \sim \{K(x,x'_n)\}$ for all $x \in X$. This clearly gives an equivalence relation on the set of all Cauchy sequences and we obtain

$$\overline{X} := \{\text{Cauchy sequences on } X\}/\sim.$$

This is a compactification of X. Actually, for $x \in X$, the constant sequence $\{x_n\}$ with $x_n = x$ for all $n \ge 0$ gives a Cauchy sequence, whence $X \subset \overline{X}$. We then obtain $\overline{X} = X \sqcup \partial X$ where $\partial X := \overline{X} \setminus X$.

The Martin kernel $K(x,y)$, which is defined on $X \times X$, is extended to $X \times \partial X$ as follows; For $x \in X$ and $\{x_n\}/\sim \in \partial X$, we define

$$K(x,\{x_n\}/\sim) := \lim_{n\to\infty} K(x,x_n).$$

(Since $\{K(x,x_n)\}$ is a Cauchy sequence in $\mathbb{R}$, the limit exists.) This is well-defined. Fix a point $y = \{y_n\}/\sim \in \partial X$. Let us write $K\delta_y(x) = K(x,y)$. Then this is always Harmonic:

$$\sum_{x\mapsto x'} P(x,x')K(x',y) = K(x,y)$$

If we take $y_1 \ne y_2$, then we have $K\delta_{y_1} \ne K\delta_{y_2}$. More generally, for any probability measure μ on the boundary ∂X, the function

$$K_\mu(x) := \int_{\partial X} K(x,y)\mu(dy)$$

is always a harmonic function.

The main theorem of the potential theory is as follows:

Theorem 2.2.1. *For every harmonic function $f \in \mathrm{Harm}(X)$, there exists a probability measure $\mu \in \mathfrak{M}_1(\partial X)$ such that $f = K_\mu$.*

We here gives some remarks. The function f is called super harmonic if $Pf \ge f$. The proof of Theorem 2.2.1 goes via showing that every super harmonic function f is of the form $f = K_\mu$ where μ is a probability measure on $\overline{X} = X \sqcup \partial X$. Note that if $f \in \mathrm{Harm}(X)_{\mathrm{ext}}$, then the corresponding measure μ has support at one point. Therefore $f = K\delta_y$ for some $y \in \partial X$. We define

$$\partial X_{\mathrm{ext}} := \left\{ y \in \partial X \,\middle|\, K\delta_y \in \mathrm{Harm}(X)_{\mathrm{ext}} \right\},$$

In generally, we have $\partial X = \partial X_{\mathrm{ext}} \sqcup \partial X_{\text{non-ext}}$. For our case, we have $\partial X = \partial X_{\mathrm{ext}}$. Now if in Theorem 2.2.1 the probability measure μ is supported on the extream points, then it is unique. Therefore, for general Markov chain (on a non-tree), we obtain the following one-to-one correspondence;

$$\begin{aligned} \mathrm{Harm}(X) &\overset{1:1}{\longleftrightarrow} \mathfrak{M}_1(\partial X_{\mathrm{ext}}) \\ K_\mu &\longleftrightarrow \mu \end{aligned}$$

This is the one-to-one correspondence stated at the beginning of this chapter.

In particular, the constant function $\mathbf{1}$ is always harmonic. The corresponding unique measure τ, supported at the extream points, is called the harmonic measure.

3

Real Beta Chain and q-Interpolation

Summary. In Sect. 3.1 we have the real analogue of the p-adic Markov chain. The state space is just $\mathbb{N}\times\mathbb{N}$. We give the probability on the arrow from $(0,0)$ to $(0,1)$ by $\frac{\alpha}{\alpha+\beta}$ and to $(1,0)$ by $\frac{\beta}{\alpha+\beta}$, respectively. If we arrive at the point (i,j), we replace α by $\alpha+2i$ and β by $\beta+2j$, respectively. Then the boundary is given by $\mathbb{P}^1(\mathbb{R})/\{\pm 1\}$. The harmonic measure is the measure $\tau^{(\alpha)\beta} = \mathrm{pr}_*\big(\tau^{\alpha}_{\mathbb{Z}_\eta}(x)\otimes\tau^{\beta}_{\mathbb{Z}_\eta}(x)\big)$, the projection of the product of two real γ-measures down to a measure on the projective line. We call this the real β-chain.

In Sect. 3.2 we introduce the q-zeta function

$$\zeta_q(s) = \frac{1}{(q^s;q)_\infty} = \prod_{n\geq 0}(1-q^{s+n})^{-1}.$$

It interpolates the p-adic $\zeta_p(s) = (1-p^{-s})^{-1}$ and the real $\zeta_\eta(s) = \Gamma\big(\frac{s}{2}\big)$. Here there is a slight problem with the real limit: we have to introduce a factor $(1-q)^s$ which destroy the periodicity of $\zeta_q(s)$. For the beta functions this problem disappear.

In Sect. 3.3 we construct the interpolation between the p-adic and the real beta chains, that is, the q-β-chain. Again the state space is $\mathbb{N}\times\mathbb{N}$. We give the probability on the arrow from $(0,0)$ to $(0,1)$ by $\frac{1-q^\beta}{1-q^{\alpha+\beta}}$ and to $(1,0)$ by $\frac{q^\beta(1-q^\alpha)}{1-q^{\alpha+\beta}}$, respectively. If we arrive at the point (i,j), we replace α by $\alpha+i$ and β by $\beta+j$, respectively. The boundary is now given by $g^{\mathbb{N}}\cup\{0\}$. Here g^i is the limit point of (i,j) as $j\to\infty$ and, the one more point, 0 is the limit point $(i,0)$ or g^i as $i\to\infty$. We see that the q-β-chain interpolate between the real and the p-adic β-chains.

3.1 Real β-Chain

We saw in Sect. 2.1 there is a one-to-one correspondence between the Markov chains on a tree and probability measures on the boundary. But there is no such correspondence on a non-tree.

Let $X = \mathbb{N}\times\mathbb{N}$ be the state space again and $X_N := \big\{(i,j)\,|\, i+j=N\big\}$. We give a Markov chain, called the real β-chain on X. Let us start from

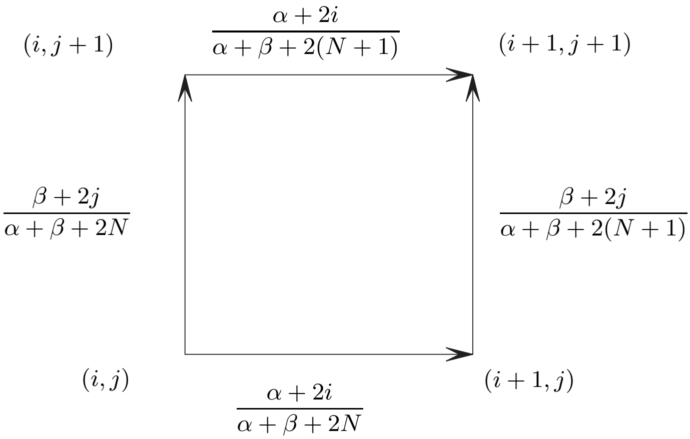


Fig. 3.1. The real β-chain

the origin $(0,0)$. We can go to $(0,1)$ and $(1,0)$ from the origin and give the probability on each walk by α and β respectably. To obtain the transition probability, we normalize by dividing by $\alpha+\beta$. If we reach the point (i,j), we replace α by $\alpha+2i$ and β by $\beta+2j$, respectively (see Fig. 3.1). Namely, if $i+j=N$, we put the probability as

$$P((i,j),(i+1,j)) := \frac{\alpha+2i}{\alpha+\beta+2N},$$
$$P((i,j),(i,j+1)) := \frac{\beta+2j}{\alpha+\beta+2N}.$$

Remark that the product of the probability from (i,j) to $(i+1,j+1)$ is same independent of which way you go. This shows that the probability does not depend on the path, but only on the initial and the final points of the path.

3.1.1 Probability Measure

To apply the potential theory, we first need the probability measure on the finite layer X_N. Let $(i,j) \in X_N$, that is, $i+j=N$. From the definition, the probability measure $\tau_N^{\alpha,\beta}(i,j)$ is given by

$$\tau_N^{\alpha,\beta}(i,j) = \tau(i,j) = (P^*)^N \delta_{(0,0)}(i,j).$$

We claim that

$$\tau_N^{\alpha,\beta}(i,j) = \frac{N!}{i!j!} \frac{\zeta_\eta(\alpha+2i, \beta+2j)}{\zeta_\eta(\alpha,\beta)} \tag{3.1}$$

To prove this, denote by $\tilde{\tau}(i,j)$ the right hand side of (3.1). Note that the probability measure τ is characterized by the following equations; $\tau(0,0) = 1$ and

$$\tau(i+1,j+1) = \tau(i,j+1)\frac{\alpha+2i}{\alpha+\beta+2(N+1)} + \tau(i+1,j)\frac{\beta+2j}{\alpha+\beta+2(N+1)}$$

Hence all we need to do is to check that $\tilde{\tau}(i,j)$ also satisfies these equations. It is clear that $\tilde{\tau}(0,0) = 1$. So let us check the second equation. From the recursion formula of the gamma function $\Gamma(s+1) = s\Gamma(s)$, we see that

$$\tilde{\tau}(i,j+1) = \tilde{\tau}(i,j)\frac{N+1}{j+1}\frac{\frac{\beta}{2}+j}{\frac{\alpha+\beta}{2}+N}, \qquad \tilde{\tau}(i+1,j) = \tilde{\tau}(i,j)\frac{N+1}{i+1}\frac{\frac{\alpha}{2}+i}{\frac{\alpha+\beta}{2}+N}$$

and

$$\tilde{\tau}(i+1,j+1) = \tilde{\tau}(i,j)\frac{(N+1)(N+2)}{(i+1)(j+1)}\frac{\left(\frac{\alpha}{2}+i\right)\left(\frac{\beta}{2}+j\right)}{\left(\frac{\alpha+\beta}{2}+N\right)\left(\frac{\alpha+\beta}{2}+N+1\right)}.$$

Hence the equation

$$\tilde{\tau}(i,j+1)\frac{\alpha+2i}{\alpha+\beta+2(N+1)} + \tilde{\tau}(i+1,j)\frac{\beta+2j}{\alpha+\beta+2(N+1)} = \tilde{\tau}(i+1,j+1)$$

is equivalent to

$$\frac{1}{j+1} + \frac{1}{i+1} = \frac{N+2}{(i+1)(j+1)}$$

and this is actually true since $i+j=N$.

Taking $\alpha=\beta=2$, we have $\tau_N^{2,2}(i,j) = \frac{1}{N+1}$ for all (i,j), $i+j=N$.

3.1.2 Green Kernel and Martin Kernel

We next want the Green kernel $G((i,j),(i',j'))$ for $(i,j) \in X_n$ and $(i',j') \in X_{n'}$. By the definition, it is clear that $G((i,j),(i',j')) = 0$ unless $i' \geq i$ and $j' \geq j$. Remembering how we defined the probability of the chain. From the property of the transition probability P, the Green kernel $G((i,j),(i',j'))$ is equal to the measure at $(i'-i, j'-j)$ with replacement α by $\alpha+2i$ and β by $\beta+2j$. Namely we have

$$G((i,j),(i',j')) = \begin{cases} \tau_{n'-n}^{\alpha+2i,\beta+2j}(i'-i,j'-j) & \text{if } i' \geq i \text{ and } j' \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Since we obtain the Green kernel, we can now get the Martin kernel. Let us write it down. The Martin kernel is given by $K((i,j),(i',j')) = \tau_{n'-n}^{\alpha+2i,\beta+2j}$ $(i'-i,j'-j)/\tau_{n'}^{\alpha,\beta}(i',j')$ for $i' \geq i$ and $j' \geq j$ (otherwise, trivially 0). One can calculate it explicitly from (3.1). Actually, it can be written as

$$\frac{\tau_{n'-n}^{\alpha+2i,\beta+2j}(i'-i,j'-j)}{\tau_{n'}^{\alpha,\beta}(i',j')}$$

$$= \frac{(n'-n)!}{(i'-i)!(j'-j)!}\frac{i'!j'!}{n'!}\frac{\zeta_\eta(\alpha+2i+2(i'-i),\beta+2j+2(j'-j))}{\zeta_\eta(\alpha+2i,\beta+2j)}\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2i',\beta+2j')}$$
$$= \frac{(n'-n)!}{(i'-i)!(j'-j)!}\frac{i'!j'!}{n'!}\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2i,\beta+2j)}.$$

Therefore we obtain

$$K((i,j),(i',j')) = \begin{cases} \dfrac{(n'-n)!}{(i'-i)!(j'-j)!}\dfrac{i'!j'!}{n'!}\dfrac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2i,\beta+2j)} & \text{if } i' \geq i \text{ and } j' \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Remark that the factor $\zeta_\eta(\alpha,\beta)/\zeta_\eta(\alpha+2i,\beta+2j)$ does not depend on i', j'.

3.1.3 Boundary

Now let us obtain the boundary of X. We first decide the Cauchy sequence. Let $\{x_n\} = \{(i_n, j_n)\}$ be a Cauchy sequence of X. By the definition it means that, for any (i,j), $\{K((i,j),(i_n,j_n))\}$ is a Cauchy sequence of $\mathbb{R}$. In particular, let us pick $(i,j) = (1,0)$ and $(0,1)$. Plugging in formula (3.3) of Martin kernel, the sequences

$$K((1,0),(i_n,j_n)) = \frac{(i_n+j_n-1)!}{(i_n-1)!j_n!}\frac{i_n!j_n!}{(i_n+j_n)!}\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2,\beta)}$$
$$= \frac{i_n}{i_n+j_n}\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2,\beta)}$$

and

$$K((0,1),(i_n,j_n)) = \frac{j_n}{i_n+j_n}\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha,\beta+2)}.$$

should be Cauchy sequences. This means that either

1. There exists some i_∞ and j_∞ such that $(i_n, j_n) \equiv (i_\infty, j_\infty)$ for all $n \gg 0$, or
2. $i_n + j_n \to \infty$ $(n \to \infty)$ and j_n/i_n converges (in wide sense) in $[0,\infty]$.

Conversely, as we will check below soon, the Martin kernel converges for all (i,j) whenever j_n/i_n converges in the wide sense. Therefore we obtain the boundary;

$$\partial X = [0,\infty] = \mathbb{P}^1(\mathbb{R})/\{\pm 1\} = \mathbb{P}^1(\mathbb{C})/\mathbb{C}^{(1)},$$

where $\mathbb{C}^{(1)} := \{x \in \mathbb{C} \,|\, |x|_\eta = 1\}$. Note that $\{\pm 1\} = \mathbb{Z}_\eta^*$.

Now assume $j_n/i_n \to |x|_\eta^2$ as $n \to \infty$ with $x \in [0,\infty]$. Then we need to show that, for all (i,j), the limit

$$K((i,j),x) := \lim_{\substack{i_n+j_n\to\infty \\ j_n/i_n \to |x|_\eta^2}} K((i,j),(i_n,j_n)).$$

exists. This is a simple calculation. Actually, from (3.3) again, we have

$$
\begin{aligned}
&K((i,j),(i_n,j_n)) \\
&\quad = \frac{(i_n+j_n-i-j)!}{(i_n+j_n)!}\frac{i_n!j_n!}{(i_n-i)!(j_n-j)!}\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2i,\beta+2j)} \\
&\quad = \frac{i_n(i_n-1)\cdots(i_n-i+1)\cdot j_n(j_n-1)\cdots(j_n-j+1)}{(i_n+j_n)(i_n+j_n-1)\cdots(i_n+j_n-i-j-1)}\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2i,\beta+2j)} \\
&\quad = \frac{\left(\frac{1}{1+\frac{j_n}{i_n}}\right)\left(\frac{1}{1+\frac{j_n+1}{i_n-1}}\right)\cdots\left(\frac{1}{1+\frac{j_n+i-1}{i_n-i+1}}\right)\cdot\left(\frac{1}{1+\frac{i_n}{j_n}}\right)\left(\frac{1}{1+\frac{i_n+1}{j_n-1}}\right)\cdots\left(\frac{1}{1+\frac{i_n+j-1}{j_n-j+1}}\right)}{1\cdot\left(1-\frac{1}{i_n+j_n}\right)\cdots\left(1-\frac{i+j-1}{i_n+j_n}\right)\frac{\zeta_\eta(\alpha+2i,\beta+2j)}{\zeta_\eta(\alpha,\beta)}}.
\end{aligned}
$$

Therefore we have

$$
\lim_{\substack{i_n+j_n\to\infty\\ j_n/i_n\to|x|_\eta^2}} K((i,j),(i_n,j_n)) = \left(\frac{1}{1+|x|_\eta^2}\right)^i\left(\frac{1}{1+|x|_\eta^{-2}}\right)^j\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2i,\beta+2j)}. \tag{3.4}
$$

This also gives the extension of the Martin kernel to the point x in the boundary $[0,\infty]$.

3.1.4 Harmonic Measure

From the calculation above we obtain the boundary $\partial X=[0,\infty]$. We next would like to calculate the harmonic measure. Remark that the harmonic measure is the measure on the boundary which corresponds to the constant function $\mathbf{1}$ (which is always harmonic). Remember that the real β-measure is given by

$$
\tau_\eta^{\alpha,\beta}(x)=\rho_\infty(x)^\alpha\rho_0(x)^\beta\frac{d^*x}{\zeta_\eta(\alpha,\beta)}.
$$

Hence, by (3.4), the Martin kernel $K((i,j),x)$ is given by

$$
K((i,j),x)=\rho_\infty(x)^{2i}\rho_0(x)^{2j}\frac{\zeta_\eta(\alpha,\beta)}{\zeta_\eta(\alpha+2i,\beta+2j)}=\frac{\tau_\eta^{\alpha+2i,\beta+2j}(x)}{\tau_\eta^{\alpha,\beta}(x)}. \tag{3.5}
$$

We claim that the real β-measure $\tau_\eta^{\alpha,\beta}$ is the harmonic measure on the boundary $\partial X=[0,\infty]$. Let us show that $K_{\tau_\eta^{\alpha,\beta}}=\mathbf{1}$. In fact, by (1.6), we have for all (i,j)

$$
\begin{aligned}
K_{\tau_\eta^{\alpha,\beta}}(i,j) &= \int_{\partial X}K((i,j),x)\tau_\eta^{\alpha,\beta}(x)=\int_0^\infty\frac{\tau_\eta^{\alpha+2i,\beta+2j}(x)}{\tau_\eta^{\alpha,\beta}(x)}\tau_\eta^{\alpha,\beta}(x) \\
&= \int_0^\infty\tau_\eta^{\alpha+2i,\beta+2j}(x)=1.
\end{aligned}
$$

Hence we obtain the claim.

3.2 q-Interpolation

There is no one-to-one correspondence between harmonic probability measures on the boundary and non-tree Markov chains. But we saw that the real β-measure is the harmonic measure of the real β-Markov chain. Both the p-adic and the real β-chains have the same state space $\mathbb{N} \times \mathbb{N}$. Here we construct a q-interpolation between them.

3.2.1 Complex β-Chain

Before we state the q-interpolation, we just make a remark about complex prime η. Let dx be the Haar measure on $\mathbb{Q}_\eta = \mathbb{C}$ (usual Lebesgue measure on $\mathbb{C}$ multiplied by two). Then we have

$$d(ax) = |a|_\eta dx, \qquad |a|_\eta = a \cdot \overline{a} = |a|^2 \qquad (a \in \mathbb{C}).$$

Let d^*x be the Haar measure on $\mathbb{C}^*$ normalized by

$$d^*x := \frac{dx}{|x|^2} \frac{1}{2\pi} = \frac{dr}{r} \frac{d\theta}{\pi} \qquad (x = re^{i\theta} \in \mathbb{C}^*)$$

Then the complex γ-integral is given by

$$\int_{\mathbb{C}^*} e^{-\pi|x|^2} |x|_\eta^s d^*x = \pi^{-s} \Gamma(s).$$

On the projective line $\mathbb{P}^1(\mathbb{C})$, we also define a metric ρ by

$$\rho((x_1 : y_1), (x_2 : y_2)) := \frac{|x_1 y_2 - y_1 x_2|_\eta}{(|x_1|_\eta^2 + |y_1|_\eta^2)(|x_2|_\eta^2 + |y_2|_\eta^2)}$$

and

$$\begin{aligned} \rho_\infty(x) &:= \rho((0:1),(1:x)) = (1+|x|^2)^{-1}, \\ \rho_0(x) &:= \rho((1:0),(1:x)) = (1+|x|^{-2})^{-1}. \end{aligned}$$

Then the complex β-measure is defied by

$$\tau_\eta^{\alpha,\beta} := \rho_\infty(x)^\alpha \rho_0(x)^\beta \frac{d^*x}{\zeta_\eta(\alpha,\beta)},$$

where $\zeta_\eta(\alpha,\beta)$ is the beta function;

$$\zeta_\eta(\alpha,\beta) := B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Then we have

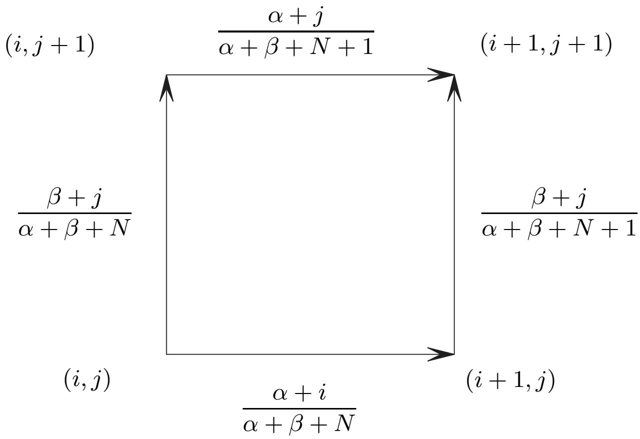


Fig. 3.2. The complex β-chain

$$\zeta_\eta(\alpha,\beta) = \int_{\mathbb{P}^1(\mathbb{C})} \rho_\infty(x)^\alpha \rho_0(x)^\beta d^*x.$$

Now we consider the complex β-chain. Let X and X_N be the same as the real β-chain. For $(i,j) \in X_N$, we give the probability by

$$P((i,j),(i+1,j)) := \frac{\alpha+i}{\alpha+\beta+N},$$
$$P((i,j),(i,j+1)) := \frac{\beta+j}{\alpha+\beta+N}.$$

see Fig. 3.2.

Remark the real beta chain is obtained by replacing α by $\alpha/2$ and β by $\beta/2$, respectively.

3.2.2 *q*-Zeta Functions

Remember the local zeta functions for a finite prime p; $\zeta_p(s) := (1-p^{-s})^{-1}$, and the complex prime η; $\zeta_\eta(s) := \Gamma(s)$. Let us compare these functions. Notice that the function $\zeta_p(s)$ is periodic with period $\frac{2\pi i}{\log p}$ and has simple poles at $\frac{2\pi i}{\log p}\mathbb{Z}$. On the other hand the function $\zeta_\eta(s)$ is "$\mathbb{Z}$-periodic", that is, $\zeta_\eta(s+1) = s\zeta_\eta(s)$ and has simple poles at $s = 0, -1, -2, \ldots$.

	finite prime p	complex prime η
local zeta function	$\zeta_p(s) := (1-p^{-s})^{-1}$	$\zeta_\eta(s) := \Gamma(s)$
periodicity	$\zeta_p\big(s+\frac{2\pi i}{\log p}\big) = \zeta_p(s)$	$\zeta_\eta(s+1) = s\zeta_\eta(s)$
poles	$\frac{2\pi i}{\log p}\mathbb{Z}$	$-\mathbb{N}$

The simplest function which has both the same periodicity and poles of the local zeta function is

$$\zeta_q(s) := \frac{1}{(q^s : q)_\infty} = \prod_{n\geq 0}(1-q^{s+n})^{-1}.$$

We call $\zeta_q(s)$ the q-zeta function. We see that the function $\zeta_q(s)$ interpolates these two local zeta. Notice that $\zeta_q(s)$ has both periodicities mentioned above. Namely, it is periodic with period $\frac{2\pi i}{\log q}$; $\zeta_q\big(s+\frac{2\pi i}{\log q}\big) = \zeta_q(s)$, and "$\mathbb{Z}$-periodic"; $\zeta_q(s+1) = (1-q^s)\zeta_q(s)$. And the poles of $\zeta_q(s)$ are located on 1/2-lattice $-\mathbb{N} + \frac{2\pi i}{\log q}\mathbb{Z}$.

Let us see that $\zeta_q(s)$ indeed gives an interpolation between $\zeta_p(s)$ and $\zeta_\eta(s)$.

1. For a finite prime $p \neq \eta$, we have

$$\lim_{N\to\infty} \zeta_{p^{-N}}\Big(\frac{s}{N}\Big) = \zeta_p(s)$$

2. For the complex prime η, the limit is not so simple. We recall the Jackson q-gamma function $\Gamma_q(s)$ defined by

$$\Gamma_q(s) := \frac{\zeta_q(s)(1-q)^{-s}}{\zeta_q(1)(1-q)^{-1}}.$$

We have just seen that the function $\zeta_q(s)$ is periodic with period $\frac{2\pi i}{\log q}\mathbb{Z}$. On the other hand the factor $(1-q)^{-s}$ is periodic with period $\frac{2\pi i}{\log(1-q)}\mathbb{Z}$. Then taking the product of this two function, we lose the periodicity. This is why we prefer to work with $\zeta_q(s)$. The q-gamma function $\Gamma_q(s)$ actually gives a q-analogue of the gamma function since

$$\Gamma_q(s) = (1-q)^{-s+1}\prod_{n\geq 0}\frac{1-q^{1+n}}{1-q^{s+n}} = \prod_{n\geq 0}\frac{1-q^{1+n}}{1-q^{s+n}}\Big(\frac{1-q^{2+n}}{1-q^{1+n}}\Big)^{s-1}.$$

Taking the limit $q \to 1$, we have from the Stirling formula

$$\begin{aligned}\lim_{q\to 1}\Gamma_q(s) &= \prod_{n\geq 0}\frac{1+n}{s+n}\Big(\frac{2+n}{1+n}\Big)^{s-1}\\ &= \lim_{N\to\infty}\frac{1\cdot 2\cdots(N-1)N}{s(s+1)\cdots(s+N-1)}(N+1)^{s-1}\\ &= \lim_{N\to\infty}\frac{\Gamma(s)\Gamma(N)}{\Gamma(s+N)}N^s \qquad\qquad (N(N+1)^{s-1}\sim N^s)\\ &= \Gamma(s).\end{aligned}$$

(Remark that the second line is the original definition of the gamma function by Euler.) This shows that

$$\lim_{q\to 1}\frac{\zeta_q(s)(1-q)^{-s}}{\zeta_q(1)(1-q)^{-1}} = \zeta_\eta(s).$$

Define the q-beta function $\zeta_q(\alpha,\beta)$ by

$$\zeta_q(\alpha,\beta) := \frac{\zeta_q(\alpha)\zeta_q(\beta)}{\zeta_q(\alpha+\beta)\zeta_q(1)} = (1-q)^{-1}\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

This function is nice since it is periodic both in α and β with period $\frac{2\pi i}{\log q}$. Note that the original beta function does not have such properties. Let us see that $\zeta_q(\alpha,\beta)$ also gives an interpolation of $\zeta_p(\alpha,\beta)$.

1. For a finite prime $p \neq \eta$, one can obtain $\zeta_p(\alpha,\beta)$ by the p-adic substitution Ⓟ$_N$;

$$(1-q)\zeta_q(\alpha,\beta)\Bigg|_{\substack{q=p^{-N}\\ \alpha=\frac{\alpha}{N},\beta=\frac{\beta}{N}}} \longrightarrow \zeta_p(\alpha,\beta) = \frac{(1-p^{-\alpha})^{-1}(1-p^{-\beta})^{-1}}{(1-p^{-\alpha-\beta})^{-1}} \quad (N\to\infty)$$

2. For the complex prime η, we have $\zeta_\eta(\alpha,\beta)$ by

$$(1-q)\zeta_q(\alpha,\beta)\Bigg|_{\substack{\text{any } q\in(0,1)\\ q=q^{\frac{1}{N}}}} \longrightarrow \zeta_\eta(\alpha,\beta) = B(\alpha,\beta) \qquad (N\to\infty)$$

and, for the real prime $\eta=\mathbb{R}$, have also $\zeta_\eta(\alpha,\beta)$ by the real substitution Ⓡ$_N$;

$$(1-q)\zeta_q(\alpha,\beta)\Bigg|_{\substack{\text{any } q\in(0,1)\\ q=q^{\frac{1}{N}},\alpha=\frac{\alpha}{2},\beta=\frac{\beta}{2}}} \longrightarrow \zeta_\eta(\alpha,\beta) = B\left(\frac{\alpha}{2},\frac{\beta}{2}\right) \qquad (N\to\infty)$$

3.3 q-β-Chain

We next consider the chain called the q-β-chain, which is also a q-interpolation between the p-adic and the real β-chains. The state space is $X=\mathbb{N}\times\mathbb{N}$ and the N-th layer X_N is also the same as in the real β-chain. For $(i,j)\in X_N$, we define the transition probability by

$$\begin{aligned} P((i,j),(i+1,j)) &:= \frac{(1-q^{\alpha+j})q^{\beta+j}}{1-q^{\alpha+\beta+i+j}}, \\ P((i,j),(i,j+1)) &:= \frac{1-q^{\beta+j}}{1-q^{\alpha+\beta+i+j}}. \end{aligned} \tag{3.6}$$

It is easy to see that the p-adic substitution Ⓟ and real substitution Ⓡ yield the (non-symmetric) p-adic and real β-chains, respectively;

$$q\text{-}\beta\text{-chain} \overset{Ⓟ_N}{\longrightarrow} \text{the } p\text{-adic } \beta\text{-chain} \qquad (N\to\infty),$$

$$q\text{-}\beta\text{-chain} \xrightarrow{\textcircled{\eta}_N} \text{the real } \beta\text{-chain} \qquad (N \to \infty).$$

In this sense, the q-β-chain interpolates between the p-adic and the real β-chain. Notice that if we take the limit $N \to \infty$ in the p-adic substitution $\textcircled{p}_N$, the probability on the walk from (i, j) to $(i+1, j)$ vanishes unless $j = 0$. This means that there are no path from (i, j) to $(i+1, j)$ unless $j = 0$ in the p-adic β-chain. Hence the form of the tree of the non-symmetric p-adic β-chain is given in Fig. 2.3. Remark that in the q-case the probability of given two points depends on the path which connects them. More precisely, we have

$$\begin{aligned} &P((i,j),(i+1,j))P((i+1,j),(i+1,j+1)) \\ &\quad = q \cdot P((i,j),(i,j+1))P((i,j+1),(i+1,j+1)). \end{aligned} \tag{3.7}$$

When q becomes 0, as in the p-adic limit, we get the vanishing of (3.7) hence the form of a tree; while when q becomes 1, as in the real limit, we get from (3.7) the independence of the probability on the path, as noted for the real β-chain.

3.3.1 q-Binomial Theorem

We want to analyze the q-β-chain and the probability measure, the Green kernel, the Martin kernel, and so on. Before that, we recall the notation of q-analogues. We denote the q-number by

$$[s]_q := \frac{1-q^s}{1-q}$$

and the q-factorial by

$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q \qquad (n \in \mathbb{N}).$$

Note that $[1]_q = 1 = [0]_q$. We define also the q-binomial coefficient by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q.$$

The fundamental properties of the q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Now an important identity is the following q-binomial theorem. Let X and Y be operators which act on some space and satisfy the q-commutativity;

$$XY = qYX. \tag{3.8}$$

Then we have

$$(X+Y)^N = \sum_{0 \le k \le N} \begin{bmatrix} N \\ k \end{bmatrix}_q Y^{N-k} X^k. \tag{3.9}$$

One can prove this formula by a simple induction on N.

3.3.2 Probability Measure

Let us consider the q-Markov chain. Notice that

$$P^* f(i,j) = \frac{(1-q^{\alpha+i-1})q^{\beta+j}}{1-q^{\alpha+\beta+i+j-1}} f(i-1,j) + \frac{(1-q^{\beta+j-1})}{1-q^{\alpha+\beta+i+j-1}} f(i,j-1).$$

This can be written as

$$P^* f(i,j) = X f(i,j) + Y f(i,j),$$

where X (resp Y) denotes the move in the x-direction (resp. y-direction) and acts on f by multiplication by the probability. From (3.7) or Fig. 3.3, we see that X and Y satisfy the q-commutativity (3.8). Now let us calculate the probability measure by applying the q-binomial theorem (3.9). The probability measure is given by

$$\tau_N^{\alpha,\beta}(i,j) = \tau(i,j) = (P^*)^N \delta_{(0,0)}(i,j)$$

for $i+j=N$. By the q-binomial theorem, we have

$$\begin{aligned}\tau_N^{\alpha,\beta}(i,j) &= (X+Y)^N \delta_{(0,0)}(i,j),\\ &= \sum_{0\le k\le N} \begin{bmatrix} N \\ k \end{bmatrix}_q Y^{N-k} X^k \delta_{(0,0)}(i,j).\end{aligned}$$

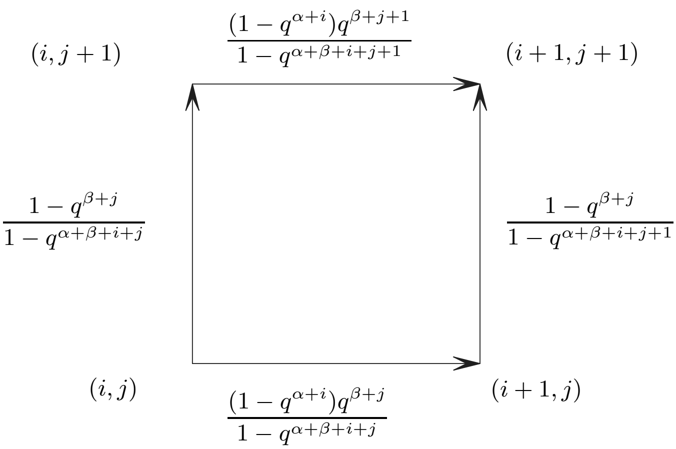


Fig. 3.3. The q-β-chain

Now X (resp. Y) acts on the delta function $\delta_{(0,0)}(i,j)$ by pushing it 1-step in the x-direction (resp. y-direction), $Y^{N-k}X^k\delta_{(0,0)}(i,j)$ has support at the point $(k, N-k)$. Hence putting $i=k$ and $j=N-k$, we have

$$= \begin{bmatrix} N \\ i \end{bmatrix}_q \frac{(1-q^{\beta+j-1})}{(1-q^{\alpha+\beta+i+j-1})} \cdots \frac{(1-q^{\beta})}{(1-q^{\alpha+\beta+i})} \frac{(1-q^{\alpha+i-1})q^{\beta}}{(1-q^{\alpha+\beta+i-1})} \cdots \frac{(1-q^{\alpha})q^{\beta}}{(1-q^{\alpha+\beta})}$$

$$= \begin{bmatrix} N \\ i \end{bmatrix}_q \frac{\zeta_q(\beta+j)}{\zeta_q(\beta)} \frac{\zeta_q(\alpha+i)}{\zeta_q(\alpha)} \frac{\zeta_q(\alpha+\beta)}{\zeta_q(\alpha+\beta+i+j)} q^{\beta i}.$$

Therefore we obtain

$$\tau_N^{\alpha,\beta}(i,j) = \begin{bmatrix} N \\ i \end{bmatrix}_q \frac{\zeta_q(\alpha+i, \beta+j)}{\zeta_q(\alpha,\beta)} q^{\beta i}. \tag{3.10}$$

3.3.3 Green Kernel and Martin Kernel

Let $i+j=n$ and $i'+j'=n$. By the similar observation at the real β-chain, the Green kernel is given by

$$G((i,j),(i',j')) = \begin{cases} \tau_{n'-n}^{\alpha+i,\beta+j}(i'-i, j'-j) & \text{if } i' \geq i \text{ and } j' \geq j, \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

and the Martin kernel is also given by $K((i,j),(i',j')) = \tau_{n'-n}^{\alpha+i,\beta+j}(i'-i, j'-j)/\tau_{n'}^{\alpha,\beta}(i',j')$ for $i' \geq i$ and $j' \geq j$ (otherwise, 0). More explicitly, we have from (3.10)

$$K((i,j),(i',j')) = \frac{\begin{bmatrix} n'-n \\ i'-i \end{bmatrix}_q \dfrac{\zeta_q(\alpha+i', \beta+j')}{\zeta_q(\alpha+i, \beta+j)} q^{(\beta+j)(i'-i)}}{\begin{bmatrix} n' \\ i' \end{bmatrix}_q \dfrac{\zeta_q(\alpha+i', \beta+j')}{\zeta_q(\alpha,\beta)} q^{\beta i'}}$$

$$= \frac{\begin{bmatrix} n'-n \\ i'-i \end{bmatrix}_q}{\begin{bmatrix} n' \\ i' \end{bmatrix}_q} \frac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i, \beta+j)} q^{-\beta i + j(i'-i)}.$$

Hence we obtain the Martin kernel;

$$K((i,j),(i',j')) = \begin{cases} \dfrac{\begin{bmatrix} n'-n \\ i'-i \end{bmatrix}_q}{\begin{bmatrix} n' \\ i' \end{bmatrix}_q} \dfrac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i, \beta+j)} q^{-\beta i + j(i'-i)} & \text{if } i' \geq i \text{ and } j' \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

3.3.4 Boundary

We next calculate the boundary. Again recall that the sequence $\{(i_n, j_n)\}$ is a Cauchy sequence if $\{K((i,j),(i_n,j_n))\}$ is a Cauchy sequence of $\mathbb{R}$ for all (i,j). In particular, $\{K((1,0),(i_n,j_n))\}$ is a Cauchy sequence of $\mathbb{R}$. Since $K((1,0),(i_n,j_n))$ is expressed from (3.12) as

$$K((1,0),(i_n,j_n)) = \frac{\left[\begin{smallmatrix} i_n+j_n-1 \\ i_n-1 \end{smallmatrix}\right]_q}{\left[\begin{smallmatrix} i_n+j_n \\ i_n \end{smallmatrix}\right]_q} \frac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+1,\beta)} q^{-\beta} = \frac{1-q^{i_n}}{1-q^{i_n+j_n}} \Big(\frac{1-q^{\alpha+\beta}}{1-q^{\alpha}}\Big) q^{\beta},$$

this is equivalent to the fact that either one of the following holds,

1. There exists some i_∞ and j_∞ such that $(i_n, j_n) \equiv (i_\infty, j_\infty)$ for all $n \gg 0$,
2. $i_n \to \infty$ as $(n \to \infty)$.
3. There exists some i_∞ such that $i_n \equiv i_\infty$ for $n \gg 0$ and $j_n \to \infty$ as $(n \to \infty)$.

Conversely, as we shall see soon, the Martin kernel converges for all (i,j) if one of these hold. As a consequence, we obtain the boundary;

$$\overline{X} = X \sqcup \{0\} \sqcup g^{\mathbb{N}}, \qquad \partial X = \{0\} \sqcup g^{\mathbb{N}},$$

where $g^n \in g^{\mathbb{N}}$ is the limit point of the sequence $\{(n,k)\}$ as $k \to \infty$ and $\{0\}$ is the limit point of the sequence $\{(n,k_n)\}$ as $n \to \infty$, any $k_n's$ (see Fig. 3.4).

We claim that g^n converges to $\{0\}$ in the boundary, just as we have seen in the p-adic β-chain. To prove this, we need to know the extension of the

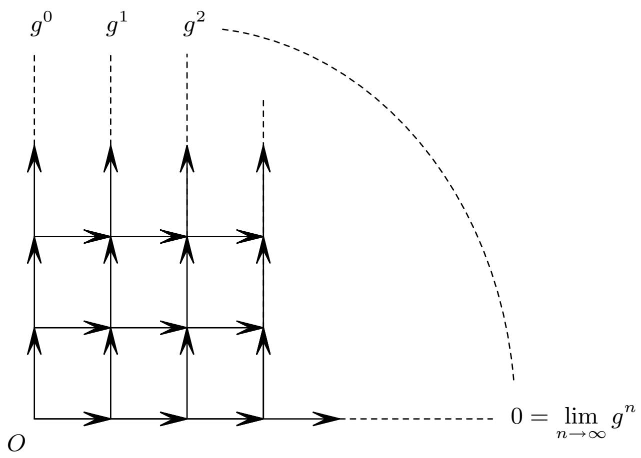


Fig. 3.4. The q-β-chain

Martin kernel to the boundary. Take $q \in (0,1)$. First of all for any $i \leq n$, we have from (3.12)

$$\begin{aligned} K((i,j),g^n) &= \lim_{k\to\infty} K((i,j),(n,k)) \\ &= \lim_{k\to\infty} \frac{\begin{bmatrix} k+n-i-j \\ n-i \end{bmatrix}_q}{\begin{bmatrix} k+n \\ n \end{bmatrix}_q} \frac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i,\beta+j)} q^{-\beta i + j(n-i)} \\ &= \frac{\zeta_q(1+n)}{\zeta_q(1+n-i)} \frac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i,\beta+j)} q^{-\beta i + j(n-i)}. \end{aligned}$$

Hence we have

$$K((i,j),g^n) = \begin{cases} \dfrac{\zeta_q(1+n)}{\zeta_q(1+n-i)} \dfrac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i,\beta+j)} q^{-\beta i + j(n-i)} & \text{if } i \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

This is an extream harmonic function which has support at $i \leq n$ and gives the extension of the Martin kernel to the boundary. But the boundary has one more point. Then we similarly calculate $K((i,j),0)$;

$$\begin{aligned} K((i,j),0) &= \lim_{n\to\infty} K((i,j),(n,k)) \\ &= \lim_{n\to\infty} \frac{\begin{bmatrix} k+n-i-j \\ n-i \end{bmatrix}_q}{\begin{bmatrix} k+n \\ n \end{bmatrix}_q} \frac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i,\beta+j)} q^{-\beta + j(n-i)}. \end{aligned}$$

Therefore we have

$$K((i,j),0) = \begin{cases} \dfrac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i,\beta)} q^{-\beta i} = \dfrac{\zeta_q(\alpha)}{\zeta_q(\alpha+i)} \dfrac{\zeta_q(\alpha+\beta+i)}{\zeta_q(\alpha+\beta)} q^{-\beta i} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

It is also easy to see that this is an extream harmonic function which is supported at $j = 0$. From (3.13) and (3.14), we have $\lim_{n\to\infty} g^n = 0$. Further, all the point in the boundary $\partial X = \{0\} \sqcup g^{\mathbb{N}}$ are extream, it holds that $\partial X = \partial X_{\text{ext}}$.

3.3.5 Harmonic Measure

Finally we give the harmonic measure of the q-β-chain. Let

$$\tau_q^{\alpha,\beta}(g^n) := \frac{\zeta_q(\alpha+n)}{\zeta_q(1+n)} \frac{q^{\beta n}}{\zeta_q(\alpha,\beta)}. \tag{3.15}$$

We call $\tau_q^{\alpha,\beta}(g^n)$ the q-β-measure. This is the probability measure on the boundary. Remark that the measure $\tau_q^{\alpha,\beta}$ has support at $g^{\mathbb{N}}$ and this is the open and dense subset of ∂X. We claim that this is the harmonic measure of the q-β-chain. First of all, for all $i \le n$, we have from (3.13)

$$\begin{aligned}\frac{\tau_q^{\alpha+i,\beta+j}(g^{n-i})}{\tau_q^{\alpha,\beta}(g^n)} &= \frac{\zeta_q(\alpha+n)}{\zeta_q(1+n-i)}\frac{\zeta_q(1+n)}{\zeta_q(\alpha+n)}\frac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i,\beta+j)}q^{(\beta+j)(n-i)-\beta n}\\ &= \frac{\zeta_q(1+n)}{\zeta_q(1+n-i)}\frac{\zeta_q(\alpha,\beta)}{\zeta_q(\alpha+i,\beta+j)}q^{-\beta i+j(n-i)}\\ &= K((i,j),g^n).\end{aligned}$$

Therefore we have

$$\begin{aligned}K_{\tau^{\alpha,\beta}}(i,j) &= \int_{\partial X} K((i,j),y)\tau_q^{\alpha,\beta}(y) = \sum_{n\ge i}\frac{\tau_q^{\alpha+i,\beta+j}(g^{n-i})}{\tau_q^{\alpha,\beta}(g^n)}\tau_q^{\alpha,\beta}(g^n)\\ &= \sum_{n\ge 0}\tau_q^{\alpha+i,\beta+j}(g^n) = 1.\end{aligned}$$

By the definition, this shows that the q-β-measure is the harmonic measure. We check the last equality. Namely,

$$\text{The } q\text{-beta sum:} \qquad \sum_{n\ge 0}\tau_q^{\alpha,\beta}(g^n) = \sum_{n\ge 0}\frac{\zeta_q(\alpha+n)}{\zeta_q(1+n)}\frac{q^{\beta n}}{\zeta_q(\alpha,\beta)} = 1. \tag{3.16}$$

This will be the most basic q-summation. It is sufficient to show that

$$\zeta_q(\alpha,\beta) = \sum_{n\ge 0}\frac{\zeta_q(\alpha+n)}{\zeta_q(1+n)}q^{\beta n}.$$

Remember that

$$\zeta_q(\alpha,\beta) = \frac{\zeta_q(\alpha)\zeta_q(\beta)}{\zeta_q(\alpha+\beta)\zeta_q(1)}.$$

Hence it is equivalent to show that

$$\begin{aligned}\frac{\zeta_q(\beta)}{\zeta_q(\alpha+\beta)} &= \sum_{n\ge 0}\frac{\zeta_q(\alpha+n)}{\zeta_q(\alpha)}\frac{\zeta_q(1)}{\zeta_q(1+n)}q^{\beta n}\\ &= \sum_{n\ge 0}\frac{(1-q^\alpha)(1-q^{\alpha+1})\cdots(1-q^{\alpha+n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}q^{\beta n} =: Z(\alpha,\beta).\end{aligned}$$

Note that

$$\begin{aligned}Z(\alpha,\beta)-q^\alpha Z(\alpha,\beta+1) &= \sum_{n\ge 0}\frac{(1-q^\alpha)(1-q^{\alpha+1})\cdots(1-q^{\alpha+n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}q^{\beta n}(1-q^{\alpha+n})\\ &= (1-q^\alpha)\sum_{n\ge 0}\frac{(1-q^{\alpha+1})(1-q^{\alpha+1})\cdots(1-q^{\alpha+n})}{(1-q)(1-q^2)\cdots(1-q^n)}q^{\beta n}\end{aligned}$$

$$= (1 - q^{\alpha}) Z(\alpha + 1, \beta)$$

and similarly

$$\begin{aligned} Z(\alpha,\beta) - Z(\alpha,\beta+1) &= \sum_{n\geq 0} \frac{(1-q^{\alpha})(1-q^{\alpha+1})\cdots(1-q^{\alpha+n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} q^{\beta n}(1-q^n) \\ &= (1-q^{\alpha}) \sum_{n\geq 1} \frac{(1-q^{\alpha+1})(1-q^{\alpha+1})\cdots(1-q^{\alpha+n-1})}{(1-q)(1-q^2)\cdots(1-q^{n-1})} q^{\beta n} \\ &= (1-q^{\alpha}) Z(\alpha+1,\beta) q^{\beta}. \end{aligned}$$

From these two equation, we have

$$Z(\alpha,\beta) - Z(\alpha,\beta+1) = \big(Z(\alpha,\beta) - q^{\alpha} Z(\alpha,\beta+1)\big) q^{\beta}.$$

Therefore we inductively obtain

$$\begin{aligned} Z(\alpha,\beta) &= Z(\alpha,\beta+1) \frac{1-q^{\alpha+\beta}}{1-q^{\beta}} \\ &= Z(\alpha,\beta+2) \frac{1-q^{\alpha+\beta+1}}{1-q^{\beta+1}} \frac{1-q^{\alpha+\beta}}{1-q^{\beta}} \\ &= \cdots \\ &= Z(\alpha,\infty) \frac{\zeta_q(\beta)}{\zeta_q(\alpha+\beta)}. \end{aligned}$$

By the definition it is obvious that $Z(\alpha,\infty) = 1$, whence we obtain the result

$$Z(\alpha,\beta) = \frac{\zeta_q(\beta)}{\zeta_q(\alpha+\beta)}.$$

This completes the proof of (3.16).

4

Ladder Structure

Summary. In Sect. 4.1 we study the ladder structure for Markov chains on trees. In Sect. 4.2 we specialize to the p-adic β-chain. On the N-th layer of the p-adic β-chain $X_N := \{(i,j) \in \mathbb{N} \times \mathbb{N} \,|\, i+j = N\}$ which is identified with $\mathbb{P}^1(\mathbb{Z}/p^N)/(\mathbb{Z}/p^N)^* \ltimes \mathbb{Z}/p^N$, we get the Hilbert space $H^{(\alpha)\beta}_{p(N)}$ of dimension $N+1$. Each space embeds unitarily into the next stage and all of them embed unitarily into the boundary space $H^{(\alpha)\beta}_p = \ell^2\big(\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p, \tau_p^{(\alpha)\beta}\big)$. This is because of the tree structure of the p-adic β-chain. On each next stage, we obtain the new function $\varphi^{(\alpha)\beta}_{p(N),m}$ of the N-th layer which is orthogonal to all the function of the $N-1$-th layer. These functions give an orthogonal basis of the N-th layer and, in the boundary space, they also gives an orthogonal basis.

In Sect. 4.2 (resp. 4.4) we give the "ladder structure" for the q-β (resp. real β) chains. In the case of the q-β and the real β-chain, we do not have a tree (we have loops). We have an embedding of the $N-1$-th layer space into the N-th layer space but it is not a unitary embedding. So we need something else. We introduce the following difference operator

$$D\varphi(i,j) := \frac{\varphi(i+1,j) - \varphi(i,j+1)}{q^i(1-q)}.$$

We consider the operator as the map on the N-th space $H^{(\alpha)\beta}_{q(N)} = \ell^2\big(X_N, \tau^{(\alpha)\beta}_{q(N)}\big)$ to $H^{(\alpha+1)\beta+1}_{q(N-1)}$. Note that both parameters α and β go up by 1 and N goes down by 1, respectively. We denote its adjoint by D^* and then obtain the ladder

$$H^{(\alpha)\beta}_{q(N)} \underset{D_N^+}{\overset{D_N}{\rightleftarrows}} H^{(\alpha+1)\beta+1}_{q(N-1)} \underset{D_{N-1}^+}{\overset{D_{N-1}}{\rightleftarrows}} H^{(\alpha+2)\beta+2}_{q(N-2)}.$$

Here the operator D^+ is a constant multiple of the adjoint D^*. One can check that the operators D^+ and D satisfy the relation

$$D_N D_N^+ - D_{N-1}^+ D_{N-1} = (\text{const.}) \cdot \mathrm{id}_{H^{(\alpha+1)\beta+1}_{q(N-1)}}.$$

We call it the Heisenberg relation up the ladder. It follows immediately that if we just take the constant function $\mathbf{1}$, going up the ladder m step, in other word, acting with D^+ m times on $\mathbf{1}$, gives an orthogonal basis of $H^{(\alpha)\beta}_{q(N)}$;

$$\varphi^{(\alpha)\beta}_{q(N),m} := (D^*)^m \mathbf{1}_{H^{(\alpha+m)\beta+m}_{q(N-m)}}, \qquad m = 0, 1, \ldots, N.$$

Taking the limit $N \to \infty$, we get the boundary space denoted by $H^{(\alpha)\beta}_q$. Now the difference operator D is defined by

$$D\varphi(g^i) := \frac{\varphi(g^{i+1}) - \varphi(g^i)}{q^i(1-q)}$$

and again we obtain its adjoint D^+ and the ladder;

$$H^{(\alpha)\beta}_q \underset{D^+}{\overset{D}{\rightleftarrows}} H^{(\alpha+1)\beta+1}_q \underset{D^+}{\overset{D}{\rightleftarrows}} H^{(\alpha+2)\beta+2}_q .$$

The operators D and D^+ again satisfy the Heisenberg relation up the ladder, and the constant function $\mathbf{1}$ also gives the orthogonal basis $\varphi^{(\alpha)\beta}_{q,m}$, $m \geq 0$, by going m step up the ladder. One checks that the Martin kernel K gives the compatibility between this finite difference operator and the operator on the boundary. Hence we have the diagonalization of the Martin kernel. Note that it is not an orthogonal projection (as in the p-adic case) because the norm of $\varphi^{(\alpha)\beta}_{q,m}$ is different from the norm of $\varphi^{(\alpha)\beta}_{q(N),m}$.

$$\begin{array}{ccccccc}
\varphi^{(\alpha)\beta}_{q,m} \in & H^{(\alpha)\beta}_q & \underset{D^+}{\overset{D}{\rightleftarrows}} & H^{(\alpha+1)\beta+1}_q & \ni \varphi^{(\alpha+1)\beta+1}_{q,m} \\
\big\downarrow K^{(\alpha)\beta}_{q(N)} & \big\downarrow K^{(\alpha)\beta}_{q(N)} & & \big\downarrow K^{(\alpha+1)\beta+1}_{q(N-1)} & \big\downarrow K^{(\alpha+1)\beta+1}_{q(N-1)} \\
\varphi^{(\alpha)\beta}_{q(N),m} \in & H^{(\alpha)\beta}_{q(N)} & \underset{D_N^+}{\overset{D_N}{\rightleftarrows}} & H^{(\alpha+1)\beta+1}_{q(N-1)} & \ni \varphi^{(\alpha+1)\beta+1}_{q(N-1),m}
\end{array}$$

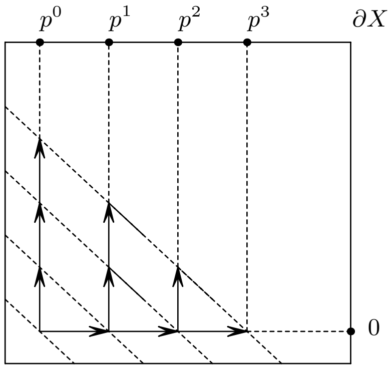
Taking the p-adic limit ⓟ, we see that the basis $\varphi^{(\alpha)\beta}_{q,m}$ converges to the basis $\varphi^{(\alpha)\beta}_{p,m}$ which we have obtained before from the tree structure. Similarly the q-basis $\varphi^{(\alpha)\beta}_{q(N),m}$ converges in the p-adic and the real limits to $\varphi^{(\alpha)\beta}_{p(N),m}$ and $\varphi^{(\alpha)\beta}_{\eta(N),m}$, respectively. We call the basis $\varphi^{(\alpha)\beta}_{q(N)}$ of the finite layer the q-Hahn polynomial and of the boundary the q-little Jacobi polynomial.

In Sect. 4.3 the ladder structure of the q-γ-chain is obtained from the q-β-chain by taking the limit $\alpha \to \infty$. We have the ladder

$$H^{\beta}_{q(N)} \underset{D_N^+}{\overset{D_N}{\rightleftarrows}} H^{\beta+1}_{q(N-1)} \underset{D_{N-1}^+}{\overset{D_{N-1}}{\rightleftarrows}} H^{\beta+2}_{q(N-2)} ,$$

where $H^{\beta}_{q(N)}$ is the space of the N-th layer of the q-γ-chain and D and D^+ are finite difference operator. The ladder satisfies the Heisenberg relation up the ladder. Hence we again obtain the basis

$$\varphi^{\beta}_{q(N),m} = (-1)^m \frac{q^{\frac{m(m-1)}{2}}}{[m]_q!} (D^+)^m \mathbf{1}_{H^{\beta+m}_{q(N-m)}} .$$



The p-adic β-chain

$X_N = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i+j=N\}$

Hilbert space $H^{(\alpha)\beta}_{p(N)} = \ell^2\big(X_N, \tau^{(\alpha)\beta}_{p(N)}\big)$

measure $\tau^{(\alpha)\beta}_{p(N)}$

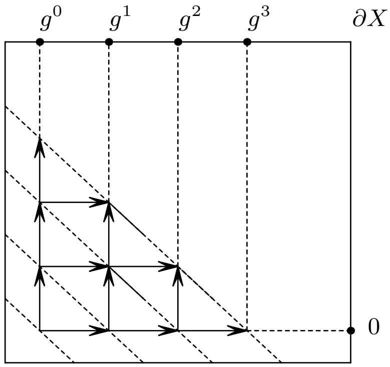
orthogonal basis $\varphi^{(\alpha)\beta}_{p(N),m}$ (p-Hahn basis)

$\partial X = \mathbb{P}^1(\mathbb{Q}_p))/\mathbb{Z}_p \ltimes \mathbb{Z}_p^* \simeq p^{\mathbb{N}} \cup \{0\}$

Hilbert space $H^{(\alpha)\beta}_{p} = \ell^2\big(\partial X, \tau^{(\alpha)\beta}_{p}\big)$

measure $\tau^{(\alpha)\beta}_{p}$

orthogonal basis $\varphi^{(\alpha)\beta}_{p,m}$ (p-Jacobi basis)



The q-β-chain

$X_N = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i+j=N\}$

Hilbert space $H^{(\alpha)\beta}_{q(N)} = \ell^2\big(X_N, \tau^{(\alpha)\beta}_{q(N)}\big)$

measure $\tau^{(\alpha)\beta}_{q(N)}$

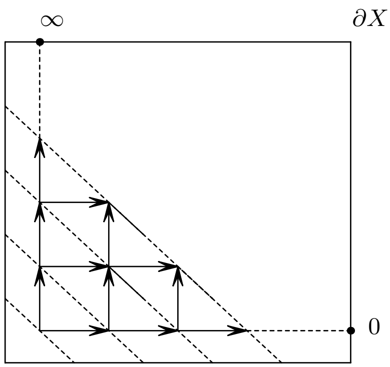
orthogonal basis $\varphi^{(\alpha)\beta}_{q(N),m}$ (q-Hahn basis)

$\partial X = g^{\mathbb{N}} \cup \{0\}$

Hilbert space $H^{(\alpha)\beta}_{q} = \ell^2\big(\partial X, \tau^{(\alpha)\beta}_{q}\big)$

measure $\tau^{(\alpha)\beta}_{q}$

orthogonal basis $\varphi^{(\alpha)\beta}_{q,m}$ (q-Jacobi basis)



The real β-chain

$X_N = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i+j=N\}$

Hilbert space $H^{\alpha,\beta}_{\eta(N)} = \ell^2\big(X_N, \tau^{\alpha,\beta}_{\eta(N)}\big)$

measure $\tau^{\alpha,\beta}_{\eta(N)}$

orthogonal basis $\varphi^{\alpha,\beta}_{\eta(N),m}$ (Hahn polynomial)

$\partial X = [0,1] = \mathbb{P}^1(\mathbb{R})/\{\pm 1\} \simeq \mathbb{P}^1(\mathbb{C})/\mathbb{C}^{(1)}$

Hilbert space $H^{\alpha,\beta}_{\eta} = L^2\big(\partial X, \tau^{\alpha,\beta}_{\eta}\big)$

measure $\tau^{\alpha,\beta}_{\eta}$

orthogonal basis $\varphi^{\alpha,\beta}_{\eta,m}$ (Jacobi polynomial)

Here $(-1)^m q^{\frac{m(m-1)}{2}}/[m]_q!$ is a normalization constant. We call this finite q-Laguerre basis. When we take the limit $N \to \infty$, we have on the boundary the similar ladder:

$$H_q^\beta \underset{D^*}{\overset{D}{\rightleftarrows}} H_q^{\beta+1} \underset{D^*}{\overset{D}{\rightleftarrows}} H_q^{\beta+2}\ ,$$

where

$$D\varphi(g^i) = \frac{\varphi(g^i) - \varphi(g^{i+1})}{q^i(1-q)},$$
$$D^*\varphi(g^i) = \frac{1}{1-q}\big(\varphi(g^i) - (1-q^i)q^{-\beta}\varphi(g^{i-1})\big).$$

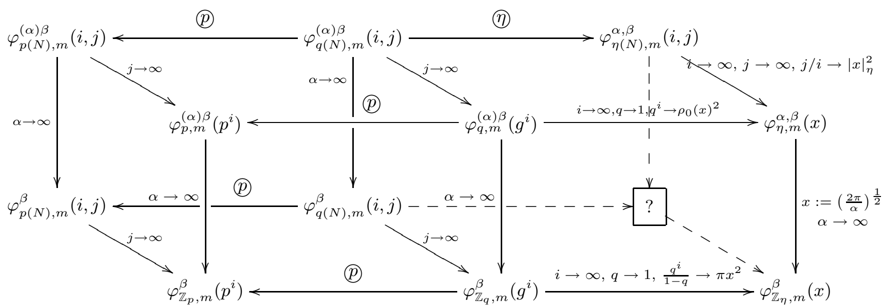
One can check that $\varphi^\beta_{q(N),m}$ gives the p-adic basis in the p-adic limit ⓟ. We denote the boundary space by $H^\beta_{\mathbb{Z}_q} := \ell^2\big(\partial X, \tau^\beta_{\mathbb{Z}_q}\big)$ where $\partial X = g^{\mathbb{N}} \cup \{0\}$ and $\tau^\beta_{\mathbb{Z}_q}$ is the q-γ-measure given by

$$\tau^\beta_{\mathbb{Z}_q}(g^i) = \frac{\zeta_q(1)}{\zeta_q(1+i)}\frac{q^{\beta i}}{\zeta_q(\beta)}.$$

In Sect. 4.4 we take the real limit of this q-ladder-structure. On the finite layers we still have difference operators D_N, D_N^+, but on the boundary we obtain differential operators D, D^+. The Heisenberg relations up the ladder are preserved. We obtain basis for the finite layers and the boundary Hilbert spaces. These are the classical Hahn polynomial and the classical Jacobi polynomial.

In Sect. 4.5 we study the Laguerre basis. We do not have a real γ-chain. The $\alpha \to \infty$ limit of the real β-chain, or the real limit of the q-γ-chain, both give the trivial chain $\mathbb{N}$ with $\mathrm{Prob}(i \mapsto i+1) = 1$, and with only one point as boundary. Nevertheless, the boundary ladder structure of the q-γ-chain converge in the real limit, and similarly, a normalized $\alpha \to \infty$ limit of the boundary ladder structure of the real β-chain converge. These are equal and give the ladder structure for the real γ-spaces, giving rise to the Laguerre basis.

We summarize the relations between our different basis in one commutative diagram of limit transitions:



In Sect. 4.6 we consider the "missing" real units. In the p-adic case, we have the following simple exact sequence

$$* \longrightarrow \mathbb{Z}_p^* \longrightarrow \mathbb{Q}_p^* \xrightarrow{|\cdot|_p} p^{\mathbb{Z}} \longrightarrow *$$

and, in the real case, we have

$$* \longrightarrow \mathbb{Z}_\eta^* \longrightarrow \mathbb{Q}_\eta \xrightarrow{|\cdot|_\eta} \mathbb{R}^+ \longrightarrow *$$

Here the p-adic units $\mathbb{Z}_p^*$ is given by $\mathbb{Z}_p^* = \boldsymbol{\mu}_{p-1} e^{2p\mathbb{Z}_p}$ where $\boldsymbol{\mu}_{p-1}$ is the set of all $p-1$ root of unity. $\mathbb{Z}_p^*$ is an open subset of $\mathbb{Q}_p^*$, while $\mathbb{Z}_\eta^* = \{\pm 1\}$ if η is real, or $\mathbb{C}^{(1)} = \{z \in \mathbb{C}^* \mid |z|_\eta = 1\}$ if η is complex, and they are both closed subset of $\mathbb{Q}_\eta^*$. This shows that we are missing the part "$e^{2p\mathbb{Z}_p}$" for the real units.

An early motivation for our investigation was the desire to do an "Iwasawa theory" for the Riemann zeta function. In the usual Iwasawa theory one construct the p-adic L-functions as $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p^* = \varprojlim (\mathbb{Z}/p^N)^*$, or the Galois group $\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{p^\infty})/\mathbb{Q}) = \varprojlim \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{p^N})/\mathbb{Q})$. In the real case we would like to understand $\mathbb{R}^+$, or rather the infinitesimal neighborhood of 1 in $\mathbb{R}^+$, as similar inverse limit, but now with loops, i.e., a Markov chain. The Markov chain is obtained as the $\alpha = \beta \to \infty$ limit of the real β-chain, just as the tree chain for $\mathbb{Z}_p^* = \varprojlim (\mathbb{Z}/p^N)^*$ is the $\alpha = \beta \to \infty$ limit of the p-adic β-chain. The ladder structure is preserved and it gives as basis the elementary symmetric functions on the finite layers, and the Hermite polynomials on the boundary. We have thus "finite quantum mechanics" approximating the usual creation and annihilation operators.

4.1 Ladder for Trees

Recall the chain for a tree. Let X_N be the N-th layer and $H_N = \ell^2(X_N, \tau_N)$ the corresponding Hilbert space (see Sect. 2.1). Here

$$\tau_N(x) = P(x_0 \mapsto x_1) \cdots P(x_{N-1} \mapsto x) \quad (x \in X_N),$$

where $x_0, x_1, \ldots, x$ is the unique path from x_0 to x. We have a chain on a tree $X = \bigsqcup_{N \geq 0} X_N$ and embeddings

$$K^* : H_N \ni \varphi \longmapsto K^*\varphi \in H_{N+1},$$

where

$$K^*\varphi(x') := \varphi(x)$$

for the unique point $x \in X_n$ such that $x \mapsto x'$. Further we have also the orthogonal projection

$$K : H_{N+1} \ni \tilde{\varphi} \longmapsto K\tilde{\varphi} \in H_N,$$

where

$$K\tilde{\varphi}(x) := \sum_{x \mapsto x'} P(x \mapsto x')\tilde{\varphi}(x')$$

Note that K^* is the adjoint operator of K since

$$\begin{aligned}(\tilde{\varphi}, K^*\varphi)_{H_{N+1}} &= \sum_{x'} \tilde{\varphi}(x')\overline{K^*\varphi(x')}\tau_{N+1}(x') \\ &= \sum_{x'} \sum_{x \mapsto x} \tilde{\varphi}(x')\overline{\varphi(x)}\tau_N(x)P(x \mapsto x') \\ &= \sum_{x} \overline{\varphi(x)}\Big(\sum_{x \mapsto x'} P(x \mapsto x')\tilde{\varphi}(x')\Big)\tau_N(x) = (K\tilde{\varphi}, \varphi)_{H_N}.\end{aligned}$$

Here we use the equation $\tau_{N+1}(x') = \tau_N(x)P(x \mapsto x')$ with $x \mapsto x'$. Hence we obtain the ladder

$$H_N \underset{K}{\overset{K^*}{\rightleftarrows}} H_{N+1} \; .$$

Note that

$$\begin{aligned}K^*\varphi(x') &= \sum_{x} \varphi(x)K(x,x')\tau_N(x), \\ K\tilde{\varphi}(x) &= \sum_{x'} \tilde{\varphi}(x')K(x,x')\tau_{N+1}(x'),\end{aligned}$$

where $K(x,x')$ is the Martin kernel;

$$K(x,x') = \begin{cases} \dfrac{1}{\tau_N(x)} & \text{if } x \mapsto x', \\ 0 & \text{otherwise,} \end{cases} = \frac{P(x \mapsto x')}{\tau_{N+1}(x')}.$$

Therefore these maps K^* and K are given in terms of the Martin kernel.

On the boundary ∂X of X, we have also the Hilbert space $H = \ell^2(\partial X, \tau)$ where $\tau \in \mathfrak{M}_1(\partial X)$. Then there is an embedding

$$K_N^* : H_N \ni \varphi \longmapsto K_N^*\varphi \in H,$$

where

$$K_N^*\varphi(\{x_n\}) := \varphi(x_N)$$

and also the orthogonal projection

$$K_N : H \ni \tilde{\varphi} \longmapsto K_N\tilde{\varphi} \in H_N,$$

where

$$K_N\tilde{\varphi}(x) := \frac{1}{\tau_N(x)} \int_{I(x)} \tilde{\varphi}(\tilde{x})\tau(\tilde{x}),$$

where $I(x) = \{\tilde{x} = \{\tilde{x}_n\} \in \partial X \,|\, \tilde{x}_N = x\}$. Then K_N^* is also adjoint of K because

$$\begin{aligned}(\tilde{\varphi}, K_N^*\varphi)_H &= \int_{\partial X} \tilde{\varphi}(\tilde{x})\overline{K_N^*\varphi(\tilde{x})}\tau(\tilde{x}) \\ &= \sum_{x_N} \overline{\varphi(x_N)} \int_{I(x_N)} \tilde{\varphi}(\tilde{x})\tau(\tilde{x}) \\ &= \sum_{x_N} \overline{\varphi(x_N)} K_N\tilde{\varphi}(x_N)\tau_N(x_N) = (K_N\tilde{\varphi}, \varphi)_{H_N}.\end{aligned}$$

Then we have also the ladder

$$H_N \underset{K_N}{\overset{K_N^*}{\rightleftarrows}} H.$$

These can be also written as

$$\begin{aligned}K_N^*\varphi(\tilde{x}) &= \sum_{x \in X_N} \varphi(x)K(x,\tilde{x})\tau_N(x), \\ K_N\tilde{\varphi}(x) &= \int_{\partial X} \tilde{\varphi}(\tilde{x})K(x,\tilde{x})\tau(\tilde{x}),\end{aligned}$$

where $K(x, \{x_N\})$ is the Martin kernel;

$$K(x, \{x_N\}) = \begin{cases} \dfrac{1}{\tau_N(x)} & \text{if } x = x_N, \\ 0 & \text{otherwise,} \end{cases} = \lim_{N\to\infty} K(x, x_N).$$

Spectrally, we have the orthogonal basis φ_m of H and $\varphi_{N,m}$ of H_N;

$$H = \bigoplus_{m \geq 0} \mathbb{C}\varphi_m, \qquad H_N = \bigoplus_{0 \leq m \leq N} \mathbb{C}\varphi_{N,m}.$$

Since K_N is the orthogonal projection, we have

$$K_N\varphi_m = \begin{cases} \varphi_{N,m} & \text{if } 0 \leq m \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

This is similar to the projection $\mathbb{Z}_p \to \mathbb{Z}/p^N$.

We remark that for the p-adic chain, the embedding $K^* : H_N \to H_{N+1}$ is unitary and the projection $K : H_{N+1} \to H_N$ is orthogonal, and similarly for the boundary space these are obtained from the Martin kernel, and reflect the projections $\mathbb{Z}/p^{N+1} \to \mathbb{Z}/p^N$ and $\mathbb{Z}_p \to \mathbb{Z}/p^N$. But in the case where we do not have a tree, K^* is not always unitary and K is not an orthogonal projection. Therefore we can not identify H_N with a subspace of H_{N+1} and also with a subspace of the boundary space H. We next see the ladder for the non-tree q-β-chain.

4.2 Ladder for the q-β-Chain

4.2.1 Finite Layer: The q-Hahn Basis

Let us consider the q-β-chain. Let $X_N = \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid i+j=N\}$. Denote by $H^{(\alpha)\beta}_{q(N)} := \ell^2(X_N, \tau^{(\alpha)\beta}_{q(N)})$ the finite N-th layer. Here $\tau^{(\alpha)\beta}_{q(N)}$ is defined in Sect. 3.2;

$$\tau^{(\alpha)\beta}_{q(N)}(i,j) = \begin{bmatrix} N \\ i \end{bmatrix}_q \frac{\zeta_q(\alpha+i,\beta+j)}{\zeta_q(\alpha,\beta)} q^{\beta i}.$$

Notice that $\tau^{(\alpha)\beta}_N(i,j)$ can be written as $\tau^{(\alpha)\beta}_N(i,j) = C^{(\alpha)\beta}_{q(N)} t^{(\alpha)\beta}_{q(N)}(i,j)$ where

$$C^{(\alpha)\beta}_{q(N)} := \frac{\zeta_q(1+N)}{\zeta_q(\alpha,\beta)\zeta_q(\alpha+\beta+N)}, \quad t^{(\alpha)\beta}_{q(N)}(i,j) := \frac{\zeta_q(\alpha+i)\zeta_q(\beta+j)}{\zeta_q(1+i)\zeta_q(1+j)} q^{\beta j}.$$

It is clear that $C^{(\alpha)\beta}_{q(N)}$ is a constant. Therefore we may concentrate on the normalized measure $t^{(\alpha)\beta}_{q(N)}$ of $\tau^{(\alpha)\beta}_{q(N)}$. For example $t^{(1)1}_{q(N)}(i,j) = q^i$ if we take $\alpha=\beta=1$.

Let $\widetilde{H}^{(\alpha)\beta}_{q(N)} := \ell^2(X_N, t^{(\alpha)\beta}_{q(N)})$ be the corresponding Hilbert space. Consider the difference

$$\nabla : \widetilde{H}^{(1)1}_{q(N)} \ni \varphi \longmapsto \nabla\varphi \in \widetilde{H}^{(1)1}_{q(N-1)},$$

where

$$\nabla\varphi(i,j) := \frac{\varphi(i,j+1)-\varphi(i+1,j)}{q^i(1-q)} \qquad ((i,j)\in X_{N-1}).$$

Then its adjoint

$$\nabla^* : \widetilde{H}^{(1)1}_{N-1} \ni \varphi \longmapsto \nabla^*\varphi \in \widetilde{H}^{(1)1}_N$$

is given by

$$\nabla^*\varphi(i,j) := \frac{\varphi(i,j-1)-\varphi(i-1,j)}{q^i(1-q)} \qquad ((i,j)\in X_N).$$

Notice that they satisfy the q-commutativity:

$$\nabla^*\nabla = q\nabla\nabla^*.$$

More important for us is that the adjoint operator ∇^* is expressed as the sum of two operator X, Y where

$$X\varphi(i,j) := -\frac{\varphi(i-1,j)}{q^i(1-q)}, \quad Y\varphi(i,j) := \frac{\varphi(i,j-1)}{q^i(1-q)}.$$

Then X and Y again satisfy the q-commutativity;

$$XY = qYX.$$

Therefore, applying the q-binomial theorem, we calculate the m-th power as follows;

$$\begin{aligned}(\nabla^*)^m \varphi(i,j) &= (X+Y)^m \varphi(i,j)\\ &= \frac{1}{\big(q^i(1-q)\big)^m} \sum_{0\le k\le m} \begin{bmatrix} m \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} \varphi(i-k, j-m+k).\end{aligned}$$

Now we make a trick. Let us consider ∇ as the operator from $\widetilde{H}^{(\alpha)\beta}_{q(N)}$ into $\widetilde{H}^{(\alpha+1)\beta+1}_{q(N-1)}$ and write this as D_N. Namely, we have the operator

$$D_N = \nabla : \widetilde{H}^{(\alpha)\beta}_{q(N)} \longrightarrow \widetilde{H}^{(\alpha+1)\beta+1}_{q(N-1)}.$$

Then its adjoint D_N^* is given by

$$D_N^* = \frac{t_q^{(1)1}(N)}{t^{(\alpha)\beta}_{q(N)}} \nabla^* \frac{t^{(\alpha+1)\beta+1}_{q(N-1)}}{t^{(1)1}_{q(N-1)}}.$$

By the simple calculation, we have explicitly

$$D_N^*\varphi(i,j) = \frac{1}{1-q}\big\{(1-q^{\alpha+i})(1-q^j)\varphi(i,j-1) - (1-q^i)(1-q^{\beta+j})q^{-\beta}\varphi(i-1,j)\big\}.$$

Hence we have the ladder

$$\widetilde{H}^{(\alpha)\beta}_{q(N)} \underset{D_N^*}{\overset{D_N}{\rightleftarrows}} \widetilde{H}^{(\alpha+1)\beta+1}_{q(N-1)} \underset{D_{N-1}^*}{\overset{D_{N-1}}{\rightleftarrows}} \widetilde{H}^{(\alpha+2)\beta+2}_{q(N-2)} \quad \cdots$$

Notice that in this ladder, upper indices go up by 1 and lower indices go down by 1.

Now we calculate the operator $[D_N D_N^* - D_{N-1}^* D_{N-1}]$ on $\widetilde{H}^{(\alpha+1)\beta+1}_{q(N-1)}$. Let $(i,j) \in X_{N-1}$. Then we have

$$\begin{aligned}&D_N D_N^* \varphi(i,j)\\ &= \frac{1}{(1-q)q^i}\big(D_N^*\varphi(i,j+1) - D_N^*\varphi(i+1,j)\big)\\ &= \frac{1}{(1-q)^2 q^i}\big\{(1-q^{\alpha+i})(1-q^{j+1})\varphi(i,j) - (1-q^i)(1-q^{\beta+j+1})q^{-\beta}\varphi(i-1,j+1)\\ &\qquad + (1-q^{i+1})(1-q^{\beta+j})q^{-\beta}\varphi(i,j) - (1-q^{\alpha+i+1})(1-q^j)\varphi(i+1,j-1)\big\}\end{aligned}$$

and

$$\begin{aligned}
&D^*_{N-1}D_{N-1}\varphi(i,j)\\
&=\frac{1}{(1-q)}\big((1-q^{\alpha+1+i})(1-q^j)D_{N-1}\varphi(i,j-1)\\
&\qquad\qquad -(1-q^i)(1-q^{\beta+1+j})q^{-\beta-1}D_{N-1}\varphi(i-1,j)\big)\\
&=\frac{1}{(1-q)^2q^i}\big\{(1-q^{\alpha+1+i})(1-q^j)\big(\varphi(i,j)-\varphi(i+1,j-1)\big)\\
&\qquad\qquad +(1-q^i)(1-q^{\beta+1+j})q^{-\beta}\big(\varphi(i,j)-\varphi(i-1,j+1)\big)\big\}.
\end{aligned}$$

Notice that in the calculation of $D^*_{N-1}D_{N-1}\varphi(i,j)$, i and j remain the same but α and β go up by 1. Now the miracle occurs. Canceling the terms $\varphi(i-1,j+1)$ and $\varphi(i+1,j-1)$, one obtains

$$\begin{aligned}
&[D_ND^*_N-D^*_{N-1}D_{N-1}]\varphi(i,j)\\
&=\varphi(i,j)\frac{1}{(1-q)^2q^i}\big\{(1-q^{\alpha+i})(1-q^{j+1})-(1-q^{\alpha+1+i})(1-q^j)\\
&\qquad\qquad +(1-q^{i+1})(1-q^{\beta+j})q^{-\beta}-(1-q^i)(1-q^{\beta+j+1})q^{-\beta}\big\}\\
&=\varphi(i,j)\frac{1}{(1-q)^2q^i}\big\{(q^j-q^{\alpha+i})(1-q)+(q^i-q^{\beta+j})q^{-\beta}(1-q)\big\}\\
&=\varphi(i,j)\frac{q^{-\beta}-q^{\alpha}}{1-q}\\
&=\varphi(i,j)[\alpha+\beta]_q q^{-\beta}.
\end{aligned}$$

Notice that the constant $[\alpha+\beta]_q q^{-\beta}$ is independent on both i and j. Therefore we obtain

$$[D_ND^*_N-D^*_{N-1}D_{N-1}]=[\alpha+\beta]_q q^{-\beta}. \tag{4.1}$$

This shows that we obtain a constant multiple of the identity operator for the difference between going down and going up in this ladder. We call this the Heisenberg relation. Note that the difference D always kill the constant function $\mathbf{1}$ and, actually, the constant function is characterized by the equation $D\mathbf{1}=0$ up to a constant multiple. Therefore one can regard D as an annihilator operator, D^* as a creation operator and the constant function $\mathbf{1}$ as the vacuum.

Define

$$\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}:=(D^*)^m\cdot\mathbf{1}_{\widetilde{H}^{(\alpha+m)\beta+m}_{q(N-m)}}=D^*_N\cdot\tilde{\varphi}^{(\alpha+1)\beta+1}_{q(N-1),m-1}\in\widetilde{H}^{(\alpha)\beta}_{q(N)}. \tag{4.2}$$

Then using the Heisenberg relation and the relation

$$[m]_q[m-1+\alpha+\beta]_q q^{-\beta-m+1}=[\alpha+\beta]_q q^{-\beta}+[m-1]_q[m+\alpha+\beta]_q q^{-\beta-m+1},$$

we inductively obtain

$$D_N\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}=D_ND^*_N\tilde{\varphi}^{(\alpha+1)\beta+1}_{q(N-1),m-1}$$

$$= \big([\alpha+\beta]_q q^{-\beta} + D^*_{N-1}D_{N-1}\big)\tilde{\varphi}^{(\alpha+1)\beta+1}_{q(N-1),m-1}$$
$$= [m]_q[m-1+\alpha+\beta]_q q^{-\beta-m+1}\tilde{\varphi}^{(\alpha+1)\beta+1}_{q(N-1),m-1}. \qquad (4.3)$$

Hence we have

$$D^*_N D_N \tilde{\varphi}^{(\alpha)\beta}_{q(N),m} = [m]_q[m-1+\alpha+\beta]_q q^{-\beta-m+1}\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}.$$

This shows that the function $\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}$ is the eigen-function of the number operator $D^*_N D_N$ with eigenvalue $[m]_q[m-1+\alpha+\beta]_q q^{-\beta-m+1}$. Note that the eigenvalues for $0 \le m \le N$ are all distinct. Therefore $\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}$ are orthogonal basis of the space $\widetilde{H}^{(\alpha)\beta}_{q(N)}$, whence of $H^{(\alpha)\beta}_{q(N)}$. We finally calculate the L^2-norm of $\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}$ as follows; Using the relation (4.3), we have

$$\begin{aligned}
||\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}||^2_{\widetilde{H}^{(\alpha)\beta}_{q(N)}} &= (\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}, \tilde{\varphi}^{(\alpha)\beta}_{q(N),m})_{\widetilde{H}^{(\alpha)\beta}_{q(N)}} \\
&= (\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}, D^*_N\tilde{\varphi}^{(\alpha+1)\beta+1}_{q(N-1),m-1})_{\widetilde{H}^{(\alpha)\beta}_{q(N)}} \\
&= (D_N\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}, \tilde{\varphi}^{(\alpha+1)\beta+1}_{q(N-1),m-1})_{\widetilde{H}^{(\alpha+1)\beta+1}_{q(N-1)}} \\
&= [m]_q[m-1+\alpha+\beta]_q q^{-\beta-m+1}||\tilde{\varphi}^{(\alpha+1)\beta+1}_{q(N-1),m-1}||^2_{\widetilde{H}^{(\alpha+1)\beta+1}_{q(N-1)}}.
\end{aligned}$$

Hence we obtain inductively

$$\begin{aligned}
||\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}||^2_{\widetilde{H}^{(\alpha)\beta}_{q(N)}} &= [m]_q! q^{-\beta m - \frac{m(m-1)}{2}} \cdot ||\mathbf{1}||^2_{\widetilde{H}^{(\alpha+m)\beta+m}_{q(N-m)}} \\
&\quad \times [m-1+\alpha+\beta]_q[m+\alpha+\beta]_q \cdots [2m+\alpha+\beta-2]_q \\
&= \frac{\zeta_q(\alpha+\beta+2m-1)}{\zeta_q(\alpha+\beta+m-1)}\Big(\frac{q^{\frac{m(m-1)}{2}}}{[m]_q!}\Big)^{-1} q^{-\beta m} \cdot (C^{(\alpha+m)\beta+m}_{N-m})^{-1}
\end{aligned} \qquad (4.4)$$

Note that, by the normalization of the measure, we have $||\mathbf{1}||^2_{\widetilde{H}^{(\alpha+m)\beta+m}_{q(N-m)}} = (C^{(\alpha+m)\beta+m}_{q(N-m)})^{-1}$.

Now we introduce the normalized basis

$$\varphi^{(\alpha)\beta}_{q(N),m} := \frac{\zeta_q(\alpha+\beta+N)}{\zeta_q(\alpha+\beta+N+m)}(-1)^m \frac{q^{\frac{m(m-1)}{2}}}{[m]_q!}\tilde{\varphi}^{(\alpha)\beta}_{q(N),m}.$$

(The factor $q^{\frac{m(m-1)}{2}}$ is essential for the p-adic limit, $\frac{\zeta_q(\alpha+\beta+N)}{\zeta_q(\alpha+\beta+N+m)}$ is essential for the real limit.) Under this normalization, from (4.2) and (4.3), we have

$$\begin{aligned}
-(1-q^{\alpha+\beta+N})^{-1}D^*_N\varphi^{(\alpha+1)\beta+1}_{q(N-1),m-1} &= [m]_q q^{1-m}\varphi^{(\alpha)\beta}_{q(N),m}, \\
-(1-q^{\alpha+\beta+N})D_N\varphi^{(\alpha)\beta}_{q(N),m} &= [m-1+\alpha+\beta]_q q^{-\beta}\varphi^{(\alpha+1)\beta+1}_{q(N-1),m-1}.
\end{aligned}$$

From the calculation (4.4), we obtain

$$||\varphi_{q(N),m}^{(\alpha)\beta}||^2_{H_{q(N)}^{(\alpha)\beta}} = \begin{bmatrix} N \\ m \end{bmatrix}_q \frac{\zeta_q(\alpha+m,\beta+m)}{\zeta_q(\alpha,\beta)} \frac{\zeta_q(\alpha+\beta+2m-1,\alpha+\beta+N)}{\zeta_q(\alpha+\beta+m-1,\alpha+\beta+N+m)} q^{-\beta m}.$$

Moreover one can calculate the explicit formula of $\varphi_{q(N),m}^{(\alpha)\beta}$ by using the q-binomial theorem as follows;

$$\varphi_{q(N),m}^{(\alpha)\beta}(i,j) = \sum_{0\le k\le m} \begin{bmatrix} i \\ m-k \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}+(k-m)\beta} \frac{\zeta_q(\alpha+i+k,\beta+j+m-k)}{\zeta_q(\alpha+i,\beta+j)}. \tag{4.5}$$

With some more calculation (q-chu-Vandermonde) we can rewrite this in a form showing more clearly the dependence on (i,j):

$$\varphi_{q(N),m}^{(\alpha)\beta}(i,j) = \frac{\zeta_q(\alpha+\beta+N)}{\zeta_q(\alpha+\beta+N+m)} \sum_{0\le k\le m} \begin{bmatrix} i \\ m-k \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q q^{ik}(-1)^k \times q^{\frac{k(k-1)}{2}+(k-m)(\beta+k)} \frac{\zeta_q(\alpha+m)\zeta_q(\beta+m)}{\zeta_q(\alpha+m-k)\zeta_q(\beta+k)}. \tag{4.6}$$

4.2.2 Boundary: The q-Jacobi Basis

Now let us extend the ladder to the boundary $\partial X = g^{\mathbb{N}} \cup \{0\}$. Remember that, for fixed i, the point $g^i \in g^{\mathbb{N}}$ is obtained by taking the limit of (i,j) as $j \to \infty$. We have already obtained the Hilbert space $H_q^{(\alpha)\beta} := \ell^2(\partial X, \tau_q^{(\alpha)\beta})$ where $\tau_q^{(\alpha)\beta}$ is the harmonic measure defined as (3.15). Here we also consider the normalized measure

$$t_q^{(\alpha)\beta}(g^i) := \zeta_q(\alpha,\beta)\tau_q^{(\alpha)\beta} = \frac{\zeta_q(\alpha+i)}{\zeta_q(1+i)} q^{\beta i}$$

and the associated Hilbert space $\widetilde{H}_q^{(\alpha)\beta} := \ell^2(\partial X, t_q^{(\alpha)\beta})$. Then we again obtain the following ladder;

$$\widetilde{H}_q^{(\alpha)\beta} \underset{D^*}{\overset{D}{\rightleftarrows}} \widetilde{H}_q^{(\alpha+1)\beta+1} \underset{D^*}{\overset{D}{\rightleftarrows}} \widetilde{H}_q^{(\alpha+2)\beta+2} .$$

Here the operator D is defined by

$$D\varphi(g^i) := \frac{\varphi(g^i)-\varphi(g^{i+1})}{q^i(1-q)}$$

and its adjoint D^* is easily calculated as

$$D^*\varphi(g^i) = [\alpha+i]_q\varphi(g^i) - [i]_q q^{-\beta}\varphi(g^{i-1}).$$

Then we have also the Heisenberg relation

$$DD^* - D^*D = [\alpha+\beta]_q q^{-\beta}\mathrm{id}_{\widetilde{H}_q^{(\alpha+1)\beta+1}}.$$

Then we get the basis $\varphi_{q,m}^{(\alpha)\beta}$ of the boundary space $\widetilde{H}_q^{(\alpha)\beta}$ defined by

$$\varphi_{q,m}^{(\alpha)\beta} := (-1)^m \frac{q^{\frac{m(m-1)}{2}}}{[m]_q!}(D^*)^m \mathbf{1}_{\widetilde{H}_q^{(\alpha+m)\beta+m}}.$$

This is also expressed as

$$\varphi_{q,m}^{(\alpha)\beta}(g^i) = \lim_{j\to\infty}\varphi_{q(i+j),m}^{(\alpha)\beta}(i,j). \tag{4.7}$$

Hence the operator D is the limit of the finite operator D_N. Remember that $\varphi_{q(i+j),m}^{(\alpha)\beta}(i,j)$ is the eigen-function of the number operator $D_N^*D_N$. Similarly to the finite case, the basis $\varphi_{q,m}^{(\alpha)\beta}(g^i)$ is also eigen-function of D^*D. In fact, by the same way to (4.2), one can obtain the following simple formulas

$$-D^*\varphi_{m-1}^{(\alpha+1)\beta+1} = [m]_q q^{1-m}\varphi_{q,m}^{(\alpha)\beta},$$
$$-D\varphi_{q,m}^{(\alpha)\beta} = [m-1+\alpha+\beta]_q q^{-\beta}\varphi_{q,m-1}^{(\alpha+1)\beta+1},$$

whence

$$D^*D\varphi_{q,m}^{(\alpha)\beta} = [m]_q[m-1+\alpha+\beta]_q q^{-\beta-m+1}\varphi_{q,m}^{(\alpha)\beta}.$$

This shows that the function $\varphi_{q,m}^{(\alpha)\beta}$ is the eigen-function of the number operator D^*D with eigenvalue $[m]_q[m-1+\alpha+\beta]_q q^{-\beta-m+1}$. By the same reason for the finite layer, $\varphi_{q,m}^{(\alpha)\beta}$ are orthogonal basis of the space $\widetilde{H}_q^{(\alpha)\beta}$, whence of $H_q^{(\alpha)\beta}$. Note that the norm of $\varphi_{q,m}^{(\alpha)\beta}$ is given as follows;

$$||\varphi_{q,m}^{(\alpha)\beta}||^2_{H_q^{(\alpha)\beta}} = \frac{\zeta_q(1)}{\zeta_q(1+m)}\frac{\zeta_q(\alpha+m,\beta+m)}{\zeta_q(\alpha,\beta)}\frac{\zeta_q(\alpha+\beta+2m-1)}{\zeta_q(\alpha+\beta+m-1)}q^{-\beta m}.$$

Remark that we can check directly that the basis $\varphi_{q(N),m}^{(\alpha)\beta}$ and $\varphi_{q,m}^{(\alpha)\beta}$ go to $\varphi_{p(N),m}^{(\alpha)\beta}$ and $\varphi_{p,m}^{(\alpha)\beta}$, respectively, if we take the p-adic substitution Ⓟ;

$$\varphi_{q(N),m}^{(\alpha)\beta}(i,j) \xrightarrow{Ⓟ} \varphi_{p(N),m}^{(\alpha)\beta}(i,j),$$
$$\varphi_{q,m}^{(\alpha)\beta}(g^i) \xrightarrow{Ⓟ} \varphi_{p,m}^{(\alpha)\beta}(p^i).$$

Also we can get the explicit formula of $\varphi_{q,m}^{(\alpha)\beta}(g^i)$ by just taking the limit (4.7). In fact, from (4.5), we have

$$
\begin{aligned}
\varphi_{q,m}^{(\alpha)\beta}(g^i) &= \sum_{0\le k\le m} \begin{bmatrix} i \\ m-k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}+(k-m)\beta} \\
&\qquad \times \frac{\zeta_q(1)}{\zeta_q(1+k)} \frac{\zeta_q(\alpha+i+k)}{\zeta_q(\alpha+i)} \\
&= \sum_{0\le k\le m} \begin{bmatrix} i \\ m-k \end{bmatrix}_q (-1)^k q^{ik} \cdot q^{\frac{k(k-1)}{2}+(k-m)(\beta+k)} \\
&\qquad \times \frac{\zeta_q(1)}{\zeta_q(1+k)} \frac{\zeta_q(\alpha+m)}{\zeta_q(\alpha+m-k)} \frac{\zeta_q(\beta+m)}{\zeta_q(\beta+k)},
\end{aligned}
\tag{4.8}
$$

whence in the limit $i \to \infty$ we pick up the $k=0$ term in the last sum,

$$
\varphi_{q,m}^{(\alpha)\beta}(0) = \lim_{i\to\infty} \varphi_{q,m}^{(\alpha)\beta}(g^i) = q^{-m\beta} \frac{\zeta_q(1)}{\zeta_q(1+m)} \frac{\zeta_q(\beta+m)}{\zeta_q(\beta)}.
$$

Some calculations gives the equation

$$
\frac{\varphi_{q,m}^{(\alpha)\beta}(g^i)}{\varphi_{q,m}^{(\alpha)\beta}(0)} = \sum_{0\le k\le m} \begin{bmatrix} m \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}-km} \frac{\zeta_q(\beta)}{\zeta_q(\beta+k)} \frac{\zeta_q(\alpha+\beta+m+k-1)}{\zeta_q(\alpha+\beta+m-1)} q^{(1+i)k}
$$

Note that this is the polynomial in q^i. Further it can be written as

$$
\frac{\varphi_{q,m}^{(\alpha)\beta}(g^i)}{\varphi_{q,m}^{(\alpha)\beta}(0)} = {}_2\phi_1(-m, \alpha+\beta+m-1, \beta; 1+i),
$$

where

$$
\begin{aligned}
&{}_r\phi_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;\gamma) \\
&\quad := \frac{\zeta_q(1)\zeta_q(\beta_1)\cdots\zeta_q(\beta_s)}{\zeta_q(\alpha_1)\cdots\zeta_q(\alpha_r)} \sum_{n\ge 0} \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{s+1-r} \\
&\qquad \frac{\zeta_q(\alpha_1+n)\cdots\zeta_q(\alpha_r+n)}{\zeta_q(1+n)\zeta_q(\beta_1+n)\cdots\zeta_q(\beta_s+n)} q^{n\gamma}.
\end{aligned}
$$

and ${}_2\phi_1(-m, \alpha+\beta+m-1, \beta; 1+i)$ becomes a polynomial in q^i, called the little q-Jacobi polynomial.

Now we have the basis of Hilbert spaces for both finite layer and for the boundary and operators between them. Therefore we obtain the following diagram:

$$
\begin{array}{ccc}
H_q^{(\alpha)\beta} & \underset{D^+}{\overset{D}{\rightleftarrows}} & H_q^{(\alpha+1)\beta+1} \\
\Big\downarrow K_{q(N)}^{(\alpha)\beta} & & \Big\downarrow K_{q(N-1)}^{(\alpha+1)\beta+1} \\
H_{q(N)}^{(\alpha)\beta} & \underset{D_N^+}{\overset{D_N}{\rightleftarrows}} & H_{q(N-1)}^{(\alpha+1)\beta+1}
\end{array}
\tag{4.9}
$$

Here $K^{(\alpha)\beta}_{q(N)} : H^{(\alpha)\beta}_q \to H^{(\alpha)\beta}_{q(N)}$ is a map on $H^{(\alpha)\beta}_q$ to the finite layer $H^{(\alpha)\beta}_{q(N)}$ defined via the Martin kernel K on $H^{(\alpha),\beta}_q$ as follows;

$$K^{(\alpha)\beta}_{q(N)}\varphi(i,j) := \int_{\partial X} K((i,j),g^n)\varphi(g^n)\tau^{\alpha,\beta}_q(g^n).$$

From (3.13) one can calculate explicitly $K^{(\alpha)\beta}_{q(N)}\varphi(i,j)$ as

$$K^{(\alpha)\beta}_{q(N)}\varphi(i,j) = \frac{1}{\zeta_q(\alpha+i,\beta+j)}\sum_{n\geq 0}\frac{\zeta_q(\alpha+i+n)}{\zeta_q(1+n)}q^{n(\beta+j)}\varphi(g^{i+n}).$$

Using the (4.2), we see that this diagram is essentially commutative in the following sense.

$$(1-q^{\alpha+\beta+N})^{-1}D^+_N \circ K^{(\alpha+1)\beta+1}_{q(N-1)} = K^{(\alpha)\beta}_{q(N)} \circ D^+,$$
$$(1-q^{\alpha+\beta+N})D_N \circ K^{(\alpha)\beta}_{q(N)} = K^{(\alpha+1)\beta+1}_{q(N-1)} \circ D.$$

This also shows that the operators D and D^+ are compatible with the Martin kernel. Immediately we have

$$K^{(\alpha)\beta}_{q(N)}\varphi^{(\alpha)\beta}_{q,m} = \begin{cases}\varphi^{(\alpha)\beta}_{q(N),m} & \text{if } 0\leq m\leq N,\\ 0 & \text{if } m > N.\end{cases}$$

Just like in the p-adic case, we have diagonalize the Martin kernel in this sense. However, remark that this is not the orthogonal projection because $||\varphi^{(\alpha)\beta}_{q,m}||_{H^{(\alpha)\beta}_q} \neq ||\varphi^{(\alpha)\beta}_{q(N),m}||_{H^{(\alpha)\beta}_{q(N)}}$. The map $K^{(\alpha)\beta}_{q(N)}$ is a q-analogue of the reduction modulo p^N of $\mathbb{Z}_p$.

4.3 Ladder for q-γ-Chain

4.3.1 Finite Layer: The Finite q-Laguerre Basis

We here recall the q-γ-chain. This is obtained by simply taking the limit $\alpha\to\infty$ of the q-β-chain. Let $H^{\beta}_{q(N)} := \ell^2(X_N, \tau^{\beta}_{q(N)})$ be the Hilbert space for the finite layer with the probability measure

$$\tau^{\beta}_{q(N)} := \lim_{\alpha\to\infty}\tau^{(\alpha)\beta}_{q(N)} = \begin{bmatrix} N \\ j \end{bmatrix}_q \frac{\zeta_q(\beta+j)}{\zeta_q(\beta)}q^{\beta i}$$

We have also the same difference operator D_N and its adjoint D^*_N between $H^{\beta}_{q(N)}$ and $H^{\beta+1}_{q(N-1)}$ by taking the limit $\alpha\to\infty$;

$$D_N\varphi(i,j) := \frac{\varphi(i,j+1)-\varphi(i+1,j)}{q^i(1-q)},$$
$$D^*_N\varphi(i,j) := \frac{1}{1-q}\{(1-q^j)\varphi(i,j-1) - (1-q^i)(1-q^{\beta+j})q^{-\beta}\varphi(i-1,j)\}$$

and hence obtain the ladder

$$H^{\beta}_{q(N)} \underset{D_N^*}{\overset{D_N}{\rightleftarrows}} H^{\beta+1}_{q(N-1)} \underset{D_{N-1}^*}{\overset{D_{N-1}}{\rightleftarrows}} H^{\beta+2}_{q(N-2)}$$

In particular it satisfies the Heisenberg relation

$$D_N D_N^* - D_{N-1}^* D_{N-1} = \frac{q^{-\beta}}{1-q}\mathrm{id}_{H^{\beta+1}_{q(N-1)}}$$

The orthogonal basis $\varphi^{\beta}_{q(N),m}$ of $H^{\beta}_{q(N)}$ is similarly defined as

$$\varphi^{\beta}_{q(N),m} := (-1)^m \frac{q^{\frac{m(m-1)}{2}}}{[m]_q!}(D^*)^m \mathbf{1}_{H^{\beta+m}_{q(N-m)}}.$$

More precisely we have from (4.5)

$$\varphi^{\beta}_{q(N),m}(i,j) = \sum_{0\le k\le m} \begin{bmatrix} i \\ m-k \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}+(k-m)\beta} \frac{\zeta_q(\beta+j+m-k)}{\zeta_q(\beta+j)}. \tag{4.10}$$

4.3.2 Boundary: The *q*-Laguerre Basis

Similarly for the boundary space $H^{\beta}_{\mathbb{Z}_q} := \ell^2(\partial X, \tau^{\beta}_{\mathbb{Z}_q})$ where

$$\tau^{\beta}_{\mathbb{Z}_q}(g^i) := \lim_{\alpha\to\infty} \tau_q^{(\alpha)\beta} = \frac{\zeta_q(1)}{\zeta_q(\beta)} \frac{q^{\beta i}}{\zeta_q(1+i)},$$

the basis $\varphi^{\beta}_{\mathbb{Z}_q,m}$ is just obtained by the limit $j\to\infty$ of $\varphi^{(\alpha)\beta}_{q(i+j),m}$ or also by the limit $\alpha\to\infty$ of $\varphi^{(\alpha)\beta}_{q,m}$:

$$\varphi^{\beta}_{\mathbb{Z}_q,m}(g^i) := \lim_{j\to\infty} \varphi^{\beta}_{q(i+j),m}(i,j) = \lim_{\alpha\to\infty} \varphi^{(\alpha)\beta}_{q,m}(g^i) \tag{4.11}$$

Of course one can construct the basis from the following difference operators;

$$D\varphi(g^i) := \frac{\varphi(g^i) - \varphi(g^{i+1})}{q^i(1-q)},$$
$$D^*\varphi(g^i) := \frac{1}{1-q}\big(\varphi(g^i) - (1-q^i)q^{-\beta}\varphi(g^{i+1})\big).$$

Then we have

$$\varphi^{\beta}_{\mathbb{Z}_q,m} = (-1)^m \frac{q^{\frac{m(m-1)}{2}}}{[m]_q!}(D^*)^m \mathbf{1}_{H^{\beta+m}_{\mathbb{Z}_q}},$$

whence from (4.8) or from (4.10)

$$\varphi^{\beta}_{\mathbb{Z}_q,m}(g^i) = \sum_{0\le k\le m} \begin{bmatrix} i \\ m-k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}+(k-m)\beta} \frac{\zeta_q(1)}{\zeta_q(1+k)}.$$

From (4.11), we have

$$\frac{\varphi^{\beta}_{\mathbb{Z}_q,m}(g^i)}{\varphi^{\beta}_{\mathbb{Z}_q,m}(0)} = {}_2\phi_1(-m,\infty,\beta;1+i).$$

Hence we obtain the following diagram between the boundary space and finite approximation.

$$\begin{array}{ccc} H^{\beta}_{\mathbb{Z}_q} & \underset{D^{+}}{\overset{D}{\rightleftarrows}} & H^{\beta+1}_{\mathbb{Z}_q} \\ {\scriptstyle K^{\beta}_{q(N)}}\downarrow & & \downarrow{\scriptstyle K^{\beta+1}_{q(N-1)}} \\ H^{\beta}_{q(N)} & \underset{D^{+}_N}{\overset{D_N}{\rightleftarrows}} & H^{\beta+1}_{q(N-1)} \end{array}$$

Here $K^{\beta}_{q(N)} : H^{\beta}_{\mathbb{Z}_q} \to H^{\beta}_{q(N)}$ is also defined via the Martin kernel;

$$\begin{aligned} K^{\beta}_{q(N)}\varphi(i,j) :&= \int_{\partial X} K((i,j),g^n)\varphi(g^n)\tau^{\beta}_{\mathbb{Z}_q}(g^n) \\ &= \frac{\zeta_q(1)}{\zeta_q(\beta+j)} \sum_{n\ge 0} \frac{q^{n(\beta+j)}}{\zeta_q(1+n)} \varphi(g^{i+n}). \end{aligned}$$

In this case, we obtain compatibility

$$\begin{aligned} D^{+}_N \circ K^{\beta+1}_{q(N-1)} &= K^{\beta}_{q(N)} \circ D^{+}, \\ D_N \circ K^{\beta}_{q(N)} &= K^{\beta+1}_{q(N-1)} \circ D. \end{aligned}$$

From this equation, we get the diagonalization of the Martin kernel

$$K^{\beta}_{q(N)}\varphi^{\beta}_{\mathbb{Z}_q,m} = \begin{cases} \varphi^{\beta}_{q(N),m} & \text{if } 0\le m\le N, \\ 0 & \text{if } m > N. \end{cases}$$

We call the basis $\varphi^{\beta}_{\mathbb{Z}_q,m}$, which is the polynomial of degree m in q^i, the q-Laguerre polynomial. By taking the limit $\alpha\to\infty$ of the formulas for $\varphi^{(\alpha)\beta}_{q,m}$, the m-th basis of $H^{(\alpha)\beta}_q$ which we call the q-Jacobi polynomial, one can obtain the corresponding formulas for $\varphi^{\beta}_{\mathbb{Z}_q,m}$. In this sense, the q-Laguerre polynomial $\varphi^{\beta}_{\mathbb{Z}_q,m}$ is a limit of the q-Jacobi polynomial.

$$\varphi^{\beta}_{\mathbb{Z}_q,m}(g^i) = \lim_{\alpha\to\infty} \varphi^{(\alpha)\beta}_{q,m}(g^i).$$

It can be also expressed as the limit of the finite q-Laguerre basis:

$$\varphi^{\beta}_{\mathbb{Z}_q,m}(g^i) = \lim_{j\to\infty} \varphi^{\beta}_{q(i+j),m}(i,j).$$

Notice that the difference operators similarly satisfies the Heisenberg relation

$$DD^* - D^*D = \frac{q^{-\beta}}{1-q}\mathrm{id}_{H^{\beta+1}_{\mathbb{Z}_q}}.$$

Hence we have the following properties

$$-D^*\varphi^{\beta+1}_{\mathbb{Z}_q,m-1} = [m]_q q^{1-m}\varphi^{\beta}_{\mathbb{Z}_q,m}$$

by the definition, and

$$-D\varphi^{\beta}_{\mathbb{Z}_q,m} = \frac{q^{-\beta}}{1-q}\varphi^{\beta+1}_{\mathbb{Z}_q,m-1}.$$

(This is obtained by the induction on m by using the Heisenberg relation.) In particular we have

$$D^*D\varphi^{\beta}_{\mathbb{Z}_q,m} = [m]_q\frac{q^{-(m+\beta-1)}}{1-q}\varphi^{\beta}_{\mathbb{Z}_q,m}.$$

This shows that $\varphi^{\beta}_{\mathbb{Z}_q,m}$ are the eigen-functions of this number operator D^*D and $[m]_q\frac{q^{-(m+\beta-1)}}{1-q}$ is the eigenvalue. Since the eigenvalues are all different for different m, this is an orthogonal basis. We have the explicit formula of this function

$$\varphi^{\beta}_{\mathbb{Z}_q,m}(g^i) = \sum_{0\le k\le m}\begin{bmatrix} i \\ m-k\end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}-(m-k)\beta+k(i-k+m)}\frac{\zeta_q(1)}{\zeta_q(1+k)}\frac{\zeta_q(\beta+m)}{\zeta_q(\beta+k)} \tag{4.12}$$

and the L^2-norm of this basis (by using the Heisenberg relation)

$$||\varphi^{\beta}_{\mathbb{Z}_q,m}||^2_{H^{\beta}_{\mathbb{Z}_q}} = \frac{\zeta_q(1)}{\zeta_q(1+m)}\frac{\zeta_q(\beta+m)}{\zeta_q(\beta)}q^{-\beta m} =: C^{\beta}_q(m).$$

Also we calculate the special value of this function at 0, that is the limit $i\to\infty$ of the value at g^i. If we take the limit $i\to\infty$, each term in (4.12) vanishes unless $k=0$. Therefore we have

$$\varphi^{\beta}_{\mathbb{Z}_q,m}(0) = \lim_{i\to\infty} \varphi^{\beta}_{\mathbb{Z}_q,m}(g^i) = C^{\beta}_q(m). \tag{4.13}$$

Let us give just two properties of this basis. First of all, one can get the formula

$$\tau^s_{\mathbb{Z}_q}(\varphi^{\beta}_{\mathbb{Z}_q,m}) = C^{\beta-s}_q(m).$$

(This gives formula (4.13) for $s=0$ since this measure for $s=0$ is the delta function.) We also have the expansion of the basis $\varphi^{\beta'}_{\mathbb{Z}_q,m}$

$$\varphi^{\beta'}_{\mathbb{Z}_q,m} = \sum_{0\le j\le m} C^{\beta'-\beta}_q(j)\varphi^{\beta}_{\mathbb{Z}_q,m-j}.$$

4.4 Ladder for η-β-Chain

4.4.1 Finite Layer: The η-Hahn Basis

In the second subsection, we saw that the basis of Hilbert space for p-adic β-chain is obtained by the p-adic limit ⓟ. Here we consider the real limit, η-β-chain. We start from the finite layer $H^{\alpha,\beta}_{\eta(N)} := \ell^2(X_N, \tau^{\alpha,\beta}_{\eta(N)})$ where $\tau^{\alpha,\beta}_{\eta(N)}$ is the probability measure given in (3.1);

$$\tau^{\alpha,\beta}_{\eta(N)}(i,j) = \binom{N}{i}\frac{\zeta_\eta(\alpha+2i,\beta+2j)}{\zeta_\eta(\alpha,\beta)}.$$

Define the difference operator $D_N : H^{\alpha,\beta}_{\eta(N)} \to H^{\alpha+2,\beta+2}_{\eta(N-1)}$ and its adjoint $D^+_N : H^{\alpha+2,\beta+2}_{\eta(N-1)} \to H^{\alpha,\beta}_{\eta(N)}$ as

$$\begin{aligned} D_N\varphi(i,j) &= \Big(\frac{\alpha+\beta}{2}+N\Big)\big(\varphi(i,j+1)-\varphi(i+1,j)\big),\\ D^+_N\varphi(i,j) &= \Big(\frac{\alpha+\beta}{2}+N\Big)^{-1}\big(j\big(\frac{\alpha}{2}+i\big)\varphi(i,j-1) - i\big(\frac{\beta}{2}+j\big)\varphi(i-1,j)\big). \end{aligned}$$

Then again we have the following ladder

$$H^{\alpha,\beta}_{\eta(N)} \underset{D^+_N}{\overset{D_N}{\rightleftarrows}} H^{\alpha+2,\beta+2}_{\eta(N-1)} \underset{D^+_{N-1}}{\overset{D_{N-1}}{\rightleftarrows}} H^{\alpha+4,\beta+4}_{\eta(N-2)}$$

Similarly to the case of the q-β-chain, it is easy to see that these operator satisfy the Heisenberg relation;

$$D_N D^+_N - D^+_{N-1}D_{N-1} = \Big(\frac{\alpha+\beta}{2}\Big)\mathrm{id}_{H^{\alpha+2,\beta+2}_{\eta(N-1)}}.$$

The orthogonal basis $\varphi^{\alpha,\beta}_{\eta(N),m}$ is obtained by

$$\varphi^{\alpha,\beta}_{\eta(N),m} := (-1)^m \frac{1}{m!}(D^+)^m \mathbf{1}_{H^{\alpha+2m,\beta+2m}_{\eta(N-m)}}.$$

It can be also given as the limit

$$\begin{aligned}\varphi^{\alpha,\beta}_{\eta(N),m}(i,j) &= \lim_{q\to 1} \varphi^{\frac{\alpha}{2},\frac{\beta}{2}}_{q(N),m}(i,j)\\ &= \sum_{0\le k\le m} \binom{i}{m-k}\binom{j}{k}(-1)^k \frac{\zeta_\eta(\alpha+2(i+k),\beta+2(j+m-k))}{\zeta_\eta(\alpha+2i,\beta+2j)}.\end{aligned}$$

We have also the following relations;

$$\begin{aligned}-D^+_N\varphi^{\alpha+2,\beta+2}_{\eta(N-1),m-1} &= m\varphi^{\alpha,\beta}_{\eta(N),m},\\ -D_N\varphi^{\alpha,\beta}_{\eta(N),m} &= \Big(m-1+\frac{\alpha+\beta}{2}\Big)\varphi^{\alpha+2,\beta+2}_{\eta(N-1),m-1}.\end{aligned}$$

Therefore we have also

$$D^+_N D_N \varphi^{\alpha,\beta}_{\eta(N),m} = m\Big(m-1+\frac{\alpha+\beta}{2}\Big)\varphi^{\alpha,\beta}_{\eta(N),m}$$

This shows that the basis $\varphi^{\alpha,\beta}_{\eta(N),m}$ is the eigen-functions of the number operator $D^+_N D_N$ with the eigenvalue $m\big(m-1+\frac{\alpha+\beta}{2}\big)$, whence an orthogonal basis of $H^{\alpha,\beta}_{\eta(N)}$. The norm of $\varphi^{\alpha,\beta}_{\eta(N),m}$ is given as follows;

$$||\varphi^{\alpha,\beta}_{\eta(N),m}||_{H^{\alpha,\beta}_{\eta(N)}} = \binom{N}{m}\frac{\zeta_\eta(\alpha+2m,\beta+2n)}{\zeta_\eta(\alpha,\beta)}\frac{\zeta_\eta(\alpha+\beta+4m-2,\alpha+\beta+2N)}{\zeta_\eta(\alpha+\beta+2m-2,\alpha+\beta+2N+2m)}.$$

4.4.2 Boundary: The η-Jacobi Basis

We next study the boundary space $H^{\alpha,\beta}_\eta := L^2(\partial X, \tau^{\alpha,\beta}_\eta)$ (see Sect. 3.1). We must treat the boundary a bit more delicately. Recall the boundary ∂X and the harmonic measure $\tau^{\alpha,\beta}_\eta$ on ∂X;

$$\partial X = [0,\infty] = \mathbb{P}^1(\mathbb{R})/\{\pm 1\}, \qquad \tau^{\alpha,\beta}_\eta(x) = \rho_\infty(x)^\alpha \rho_0(x)^\beta \frac{d^*x}{\zeta_\eta(\alpha,\beta)}.$$

Change the variable

$$\mathbb{P}^1(\mathbb{R})/\{\pm 1\} \ni x \longmapsto z \in [-1,1]; \qquad z = \rho_\infty(x)^2 - \rho_0(x)^2 = \frac{1-x^2}{1+x^2}.$$

Then it holds that

$$\rho_\infty(x)^2 = \frac{1}{1+x^2} = \frac{1+z}{2}, \qquad \rho_0(x)^2 = \frac{1}{1+x^{-2}} = \frac{1-z}{2},$$

$$d^* x = \frac{1}{2}\Big(\frac{1+z}{2}\Big)^{-1}\Big(\frac{1-z}{2}\Big)^{-1} dz$$

Then we have an isomorphism

$$H_\eta^{\alpha,\beta} \simeq L^2\Big([-1,1], \Big(\frac{1+z}{2}\Big)^{\frac{\alpha}{2}-1}\Big(\frac{1-z}{2}\Big)^{\frac{\beta}{2}-1}\frac{dz}{2\zeta_\eta(\alpha,\beta)}\Big).$$

Again we have the ladder;

$$H_\eta^{\alpha,\beta} \underset{D^+}{\overset{D}{\rightleftarrows}} H_\eta^{\alpha+2,\beta+2} \underset{D^+}{\overset{D}{\rightleftarrows}} H_\eta^{\alpha+4,\beta+4}$$

Here the operators D and D^+ are defined by

$$D = \rho_\infty(x)^{-2}\rho_0(x)^{-2}\frac{x}{2}\frac{\partial}{\partial x} = -2\frac{\partial}{\partial z},$$
$$D^+ = -\rho_\infty(x)^{-\alpha}\rho_0(x)^{-\beta}\frac{x}{2}\frac{\partial}{\partial x}\rho_\infty(x)^\alpha\rho_\infty(x)^\beta$$

and these are the limit of the operators D_N and D_N^+, respectively. One can also see that they satisfy the Heisenberg relation

$$DD^+ - D^+D = \Big(\frac{\alpha+\beta}{2}\Big)\mathrm{id}_{H_\eta^{(\alpha+2)\beta+2}}.$$

Note that D^+ is the adjoint of D up to a constant multiple. Remember that for finite layer $H_{\eta(N)}^{\alpha,\beta}$, the orthogonal basis $\varphi_{\eta(N),m}^{\alpha,\beta}$ is the eigen-function of $D_N^+ D_N$ with eigenvalue $m\big(m-1+\frac{\alpha+\beta}{2}\big)$. On the boundary space, we can also obtain the orthogonal basis $\varphi_{\eta,m}^{\alpha,\beta}$ which is the eigen-function of D^+D. Actually, let

$$\varphi_{\eta,m}^{\alpha,\beta} := (-1)^m\frac{1}{m!}(D^+)^m\mathbf{1}_{H_\eta^{\alpha+2m,\beta+2m}}.$$

Then we have

$$-D^+\varphi_{\eta,m-1}^{\alpha+2,\beta+2} = m\varphi_{\eta,m}^{\alpha,\beta},$$
$$-D\varphi_{\eta,m}^{\alpha,\beta} = \Big(m-1+\frac{\alpha+\beta}{2}\Big)\varphi_{\eta,m-1}^{\alpha+2,\beta+2},$$

whence

$$D^+D\varphi_{\eta,m}^{\alpha,\beta} = m\Big(m-1+\frac{\alpha+\beta}{2}\Big)\varphi_{\eta,m}^{\alpha,\beta}.$$

This shows that $\varphi_{\eta,m}^{\alpha,\beta}$ is the eigen-function of the number operator D^+D with the same eigenvalue as $\varphi_{\eta(N),m}^{\alpha,\beta}$, whence an orthogonal basis of $H_\eta^{\alpha,\beta}$. More precisely, it can be written in terms of two types of limits. In fact, it is the limit of the real finite approximation $\varphi_{\eta(i+j),m}^{\alpha,\beta}$;

$$\varphi_{\eta,m}^{\alpha,\beta}(x) = \lim_{\substack{i,j\to\infty \\ j/i\to x^2}} \varphi_{\eta(i+j),m}^{\alpha,\beta}(i,j). \tag{4.14}$$

On the other hand, it is expressed as the limit of the boundary function for q-β-chain;

$$\varphi_{\eta,m}^{\alpha,\beta}(x) = \lim_{\substack{q\to 1, i\to\infty \\ q^i\to\rho_0(x)^2}} \varphi_{q,m}^{\frac{\alpha}{2},\frac{\beta}{2}}(g^i). \tag{4.15}$$

Also it can be written as the function in z since $\frac{i-j}{i+j} \to z$ if $i/j \to x^2$;

$$\begin{aligned} \varphi_m^{\alpha,\beta}(z) &= \lim_{\substack{i,j\to\infty \\ \frac{i-j}{i+j}\to z}} \varphi_{\eta(i+j),m}^{\alpha,\beta}(i,j) \\ &= P_m^{\beta/2-1,\alpha/2-1}(z), \end{aligned}$$

where $P_m^{\alpha,\beta}(z)$ is the classical Jacobi polynomial of degree m. The norm is calculated as

$$||\varphi_{\eta,m}^{\alpha,\beta}||_{H_\eta^{\alpha,\beta}}^2 = \frac{\zeta_\eta(\alpha+2m,\beta+2m)}{\zeta_\eta(\alpha,\beta)} \frac{\zeta_\eta(\alpha+\beta+4m-2)}{\zeta_\eta(\alpha+\beta+2m-2)} \frac{\pi^m}{m!}.$$

Remember that $\zeta_\eta(\alpha,\beta)$ is the beta function and similarly $\zeta_\eta(\alpha)$ is the gamma function (multiplied by $\pi^{-\frac{\alpha}{2}}$).

Now we obtain the following diagram

$$\begin{array}{ccc} H_\eta^{\alpha,\beta} & \underset{D^+}{\overset{D}{\rightleftarrows}} & H_\eta^{\alpha+2,\beta+2} \\ {\scriptstyle K_{\eta(N)}^{(\alpha)\beta}}\downarrow & & \downarrow{\scriptstyle K_{\eta(N-1)}^{\alpha+2,\beta+2}} \\ H_{\eta(N)}^{\alpha,\beta} & \underset{D_N^+}{\overset{D_N}{\rightleftarrows}} & H_{\eta(N-1)}^{\alpha+2,\beta+2} \end{array}$$

where $K_{\eta(N)}^{\alpha,\beta}$ is also defined via the Martin kernel:

$$K_{\eta(N)}^{\alpha,\beta}\varphi(i,j) := \int_{\partial X} K((i,j),x)\varphi(x)\tau_\eta^{\alpha,\beta}(x).$$

From (3.5), it is represented as

$$K_{\eta(N)}^{\alpha,\beta}\varphi(i,j) = \int_{\mathbb{P}^1(\mathbb{R})} \varphi(x)\tau_\eta^{\alpha+2i,\beta+2j}(x).$$

Then we have the compatibility of the Martin kernel and D^+ and D.

$$K_{\eta(N-1)}^{\alpha+2,\beta+2} \circ D = D_N \circ K_{\eta(N)}^{\alpha,\beta},$$

$$K^{\alpha,\beta}_{\eta(N)} \circ D^+ = D^+_N \circ K^{\alpha+2,\beta+2}_{\eta(N-1)}$$

and hence

$$K^{\alpha,\beta}_{\eta(N)} \varphi^{\alpha,\beta}_{\eta,m} = \begin{cases} \varphi^{\alpha,\beta}_{\eta(N),m} & \text{if } 0 \le m \le N, \\ 0 & \text{if } m > N. \end{cases}$$

Therefore we have the diagonalization of the Martin kernel $K^{\alpha,\beta}_{\eta(N)}$ in such a sense. However, again it is not an orthogonal projection on the finite layer. But, in a sense, it is regarded as a real analogue of the p-adic reduction modulo p^N.

It is natural that one consider the real γ-theory by taking the limit as $\alpha \to \infty$. However, as we have noticed before, there is no finite layer for γ-chain. We do not know how to normalize it, whence there is no finite approximation. But on the boundary, if we take a scaling limit $\alpha \to \infty$, we can obtain the classical Laguerre polynomial and, instead of the β-measure, the γ-measure.

The limit (4.14) and (4.15) can be checked directly. More conceptually, these follow from the fact that the operator

$$D = \rho_\infty(x)^{-2}\rho_0(x)^{-2}\frac{x}{2}\frac{\partial}{\partial x} = (1+x^2)(1+x^{-2})\frac{x}{2}\frac{\partial}{\partial x}.$$

is obtained as the limit of the corresponding operator. Since the operator D^+ is just adjoint of D up to a constant multiple, we only need to look at the operator D;

1. We now calculate the limit

$$\lim_{\substack{i,j\to\infty \\ j/i\to x^2}} D_N\varphi(i,j) = \lim_{\substack{i,j\to\infty \\ j/i\to x^2}} \Big(\frac{\alpha+\beta}{2} + i + j\Big)\big(\varphi(i,j+1) - \varphi(i+1,j)\big).$$

Note that if we take the limit $i,j \to \infty$ with $j/i \to x^2$, we have

$$\frac{i-j}{i+j} \to \frac{1}{1-x^2} - \frac{1}{1+\frac{1}{x^2}} = \frac{1-x^2}{1+x^2} = z.$$

Define

$$\varphi(x) := \Phi(z) \quad \text{and set} \quad \Phi(i,j) := \Phi\Big(\frac{i-j}{i+j}\Big).$$

Then we have

$$\begin{aligned} D_N\Phi(i,j) &= \Big(\frac{\alpha+\beta}{2} + i + j\Big)\Big(\Phi\Big(\frac{i-j-1}{i+j+1}\Big) - \Phi\Big(\frac{i-j+1}{i+j+1}\Big)\Big) \\ &= -2\frac{\frac{\alpha+\beta}{2} + i + j}{i+j+1}\Bigg(\frac{\Phi\Big(\frac{i-j-1}{i+j+1}\Big) - \Phi\Big(\frac{i-j+1}{i+j+1}\Big)}{-\frac{2}{i+j+1}}\Bigg). \end{aligned}$$

Therefore one obtains

$$\lim_{\substack{i,j\to\infty \\ j/i\to x^2}} D_N\Phi(i,j) = -2\frac{\partial}{\partial z}\Phi(z).$$

On the other hand it is clear that

$$\begin{aligned}\frac{\partial}{\partial x}\varphi(x) &= \Big(\frac{\partial}{\partial z}\Phi(z)\Big)\frac{\partial}{\partial x}\Big(\frac{1-x^2}{1+x^2}\Big)\\ &= \Big(\frac{\partial}{\partial z}\Phi(z)\Big)\frac{-4x}{(1+x^2)^2}\\ &= -4\Big(\frac{\partial}{\partial z}\Phi(z)\Big)\rho_\infty(x)^2\rho_0(x)^2\frac{1}{x}.\end{aligned}$$

Hence we have

$$-2\frac{\partial}{\partial z}\Phi(z) = \rho_\infty(x)^2\rho_0(x)^{-2}\frac{x}{2}\frac{\partial}{\partial x}\varphi(x) = D\varphi(x)$$

This shows that the operator D_N converges to D as $i,j\to\infty$ with $j/i\to x^2$.

2. Another case is the limit $q\to 1$, $i\to\infty$ with $q^i\to\rho_0(x)^2$ of the difference operator $D: H_q^{(\alpha)\beta}\to H_q^{(\alpha+1)\beta+1}$,

$$D\varphi(g^i) = \frac{\varphi(g^i)-\varphi(g^{i+1})}{q^i(1-q)}.$$

Define

$$z = (1+x^{-2})^{-1} = \rho_0(x)^2, \qquad \varphi(x) := \Phi(z) \quad \text{and set} \quad \Phi(g^i) := \Phi(q^i).$$

Then we have

$$\lim_{\substack{q\to 1, i\to\infty \\ q^i\to\rho_0(x)^2}} \frac{\Phi(q^i)-\Phi(q^{i+1})}{q^i(1-q)} = \frac{\partial}{\partial z}\Phi(z).$$

On the other hand we have also

$$\begin{aligned}\frac{\partial}{\partial x}\varphi(x) &= \Big(\frac{\partial}{\partial z}\Phi(z)\Big)\frac{\partial}{\partial x}\big(1+x^{-2}\big)^{-1}\\ &= \Big(\frac{\partial}{\partial z}\Phi(z)\Big)\frac{2x}{(1+x^2)^2}\\ &= \Big(\frac{\partial}{\partial z}\Phi(z)\Big)\rho_\infty(x)^2\rho_0(x)^2\frac{2}{x},\end{aligned}$$

whence

$$\frac{\partial}{\partial z}\Phi(z) = \rho_\infty(x)^{-2}\rho_0(x)^{-2}\frac{x}{2}\frac{\partial}{\partial x}\varphi(x) = D\varphi(x)$$

4.5 The η-Laguerre Basis

Remember that we unfortunately do not have a real γ-chain. However we do have a basis on the boundary space for the real β-measure. We consider the scaled limit of $\alpha \to \infty$ of the Jacobi basis. We denote the boundary space by $H^{\beta}_{\mathbb{Z}_\eta} = L^2(\mathbb{R}/\{\pm 1\}, \tau^{\beta}_{\mathbb{Z}_\eta})$ where $\tau^{\beta}_{\mathbb{Z}_\eta}$ is the η-γ-measure defined by

$$\tau^{\beta}_{\mathbb{Z}_\eta}(x) = e^{-\pi x^2} |x|^2_\eta \frac{d^*x}{\zeta_\eta(\beta)}.$$

Note that $\zeta_\eta(\beta) = \pi^{-\frac{\beta}{2}} \Gamma\left(\frac{\beta}{2}\right)$. We have also the ladder

$$H^{\beta}_{\mathbb{Z}_\eta} \underset{D^+}{\overset{D}{\rightleftarrows}} H^{\beta+2}_{\mathbb{Z}_\eta} \underset{D^+}{\overset{D}{\rightleftarrows}} H^{\beta+4}_{\mathbb{Z}_\eta}$$

Here D is the scaled limit $\alpha \to \infty$ of the differential operator $D^{\alpha,\beta}_\eta$ which we have obtained for η-β-chain. It is given as

$$D = \lim_{\alpha\to\infty} \left(\frac{\alpha}{2}\right)^{-1} \pi\left(\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\right) \circ D^{\alpha,\beta}_\eta \circ \pi\left(\left(\frac{\alpha}{2\pi}\right)^{-\frac{1}{2}}\right),$$

where $\pi(a)\varphi(x) = \varphi(a^{-1}x)$. Now this limit is easily calculated as

$$D = \frac{1}{2\pi x}\frac{\partial}{\partial x}.$$

Then the adjoint operator $D^* = D^*_\beta$ is given by

$$D^* = \lim_{\alpha\to\infty} \pi\left(\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\right) \circ (D^+)^{\alpha,\beta}_\eta \circ \pi\left(\left(\frac{\alpha}{2\pi}\right)^{-\frac{1}{2}}\right)$$

and has a simple formula

$$D^* = e^{\pi x^2} |x|^{-\beta}_\eta \left(-\frac{x}{2}\frac{\partial}{\partial x}\right) e^{-\pi x^2} |x|^{\beta}_\eta = -\frac{x}{2}\frac{\partial}{\partial x} + \pi x^2 - \frac{\beta}{2}.$$

And again these operator D and D^* satisfy the Heisenberg relation

$$DD^*_\beta - D^*_{\beta+2} D = \mathrm{id}_{H^{\beta+2}_{\mathbb{Z}_\eta}}.$$

Using this formula, we have the eigen-function of the number operator DD^* with eigenvalue m given by

$$\varphi^{\beta}_{\mathbb{Z}_\eta,m} := (-1)^m \frac{1}{m!} (D^*)^m \mathbf{1}_{H^{\beta+2m}_{\mathbb{Z}_\eta}}.$$

This basis can be expressed in various ways. First of all, it is given by the scaled limit of the classical Jacobi polynomial;

$$\varphi^{\beta}_{\mathbb{Z}_\eta,m}(x) = \lim_{\alpha\to\infty} \varphi^{\alpha,\beta}_{\eta,m}\left(\left(\frac{\alpha}{2\pi}\right)^{-\frac{1}{2}} x\right).$$

Similarly one can write this by the limit of the q-Laguerre basis as follows:

$$\varphi^{\beta}_{\mathbb{Z}_\eta,m}(x) = \lim_{\substack{q\to 1, i\to\infty \\ \frac{q^i}{1-q}\to \pi x^2}} \varphi^{\beta}_{\mathbb{Z}_q,m}(g^i). \tag{4.16}$$

From the definition we have again a simple formula

$$\begin{aligned}\varphi^{\beta}_{\mathbb{Z}_\eta,m}(x) &= e^{\pi x^2}|x|_\eta^{2-\beta}\frac{1}{m!}\left(\frac{1}{2x}\frac{\partial}{\partial x}\right)^m e^{-\pi m^2}|x|_\eta^{\beta+2(m-1)} \\ &= \frac{\pi^m}{m!}\sum_{0\le k\le m}\binom{m}{k}(-1)^k\frac{\zeta_\eta(\beta+2m)}{\zeta_\eta(\beta+2k)}x^{2k} \\ &= (-1)^m L_m^{\frac{\beta}{2}-1}(\pi x^2),\end{aligned}$$

where $L^{\beta}_m(x)$ is the classical Laguerre polynomial. Then let us check the formula (4.16). Remember the q-Difference operator for the q-Laguerre basis;

$$D\varphi(g^i) = \frac{\varphi(g^i)-\varphi(g^{i+1})}{q^i(1-q)}$$

Take the limit $i\to\infty$, $q\to 1$ in such a way that $\frac{q^i}{1-q}\to \pi x^2$. We also put

$$\varphi(x) = \Phi(z), \quad z=\pi x^2, \quad \Phi(g^i) = \Phi\left(\frac{q^i}{1-q}\right)$$

Then we have

$$(1-q)D\Phi(z) = \frac{\Phi(z)-\Phi(qz)}{\dfrac{q^i}{1-q}-\dfrac{q^{i+1}}{1-q}} = \frac{\Phi(z)-\Phi(qz)}{z-qz} \longrightarrow \frac{\partial}{\partial z}\Phi(z)$$

as $q\to 1$. Changing the variable x to z, we have

$$\frac{\partial}{\partial x}\varphi(x) = \frac{\partial}{\partial z}\Phi(z)\frac{\partial}{\partial x}z$$

Hence we have

$$\frac{1}{2\pi x}\frac{\partial}{\partial x} = \frac{\partial}{\partial z}.$$

We finally see the limit of the special number $C_q^{\beta}(m)$. In the p-adic limit Ⓟ (note that the p-adic limit $q\to 0$ but $q^\beta \to p^{-\beta}$), we have

$$C_q^{\beta}(m) \xrightarrow{\textcircled{p}} C_p^{\beta}(m) := \begin{cases} 1 & \text{if } m=0, \\ (1-p^{-\beta})p^{\beta m} & \text{if } m>0, \end{cases}$$

Similarly in the real limit $\textcircled{\eta}$, it holds that

$$C_q^{\beta}(m) \xrightarrow{\textcircled{\eta}} C_\eta^{\beta}(m) := \frac{\pi^m}{m!}\frac{\zeta_\eta(\beta+2m)}{\zeta_\eta(\beta)} = \frac{1}{m!}\frac{\beta}{2}\left(\frac{\beta}{2}+1\right)\cdots\left(\frac{\beta}{2}+m-1\right).$$

With these values the formulas for the q-Laguerre basis at the end of 4.3.2 continue to hold in the p-adic and real limits.

4.6 Real Units

Let us now consider the limit $\alpha, \beta \to \infty$ of the p-β-chain. The "up-stares" of this chain is the tree with valencies $(p-1)$ at the root, and all other valencies equal to p. The probability is given by $1/(p-1)$ at the first stage and $1/p$ at the other stages, whence it is the random walk. Namely, each arrow gets the same probability from the same point. The boundary is $\mathbb{Z}_p^*$ and the harmonic measure is just multiplicative Haar on $\mathbb{Z}_p^*$ normalized by $d^*(\mathbb{Z}_p^*) = 1$. We consider the same thing for the real.

We take the limit $\alpha, \beta \to \infty$ with $\alpha = \beta$ in the η-β-chain. The probability of going from (i, j) to $(i+1, j)$ or $(i, j+1)$ is given by $1/2$. Then the probability measure on the N-th layer X_N is very simple. In fact it is given by the limit $\alpha = \beta \to \infty$ of the probability $\tau_{\eta(N)}^{(\alpha)\beta}$;

$$\tau_{\eta(N)}(i, j) := \lim_{\alpha\to\infty} \tau_{\eta(N)}^{(\alpha)\alpha}(i, j) = \frac{N!}{i!j!}\frac{1}{2^N}.$$

Let us obtain the harmonic measure on the boundary. At first, the Martin kernel is given as follows;

$$K((i, j), (i', j')) = \begin{cases} \dfrac{(i' + j' - (i + j))}{(i' - i)!(j' - j)!}\dfrac{i'!j'!}{(i' + j')!}2^{i+j} & \text{if } i' \geq i,\ j' \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

One can check that the sequence (i_n, j_n) in the state space converges in the Martin metric if and only if j_n/i_n converges in the wide sense in $[0, \infty] = \mathbb{P}^1(\mathbb{R})/\{\pm 1\}$. Equivalently, $\frac{i_n - j_n}{i_n + j_n}$ converges in $[-1, 1]$, that is the Cayley transform. We also extend the Martin kernel to the boundary by

$$K((i, j), x) = \lim_{j'/i' \to |x|_\eta^2} K((i, j), (i', j')) = 2^{i+j}\rho_\infty(x)^{2i}\rho_0(x)^{2j}$$

for $x \in [0, \infty]$ or, by the Cayley transform,

$$K((i, j), z) = \lim_{\frac{i'-j'}{i'+j'} \to z} K((i, j), (i', j')) = (1 + z)^i(1 - z)^j$$

for $z \in [-1, 1]$. Here

$$z = \rho_\infty(x)^2 - \rho_0(x)^2 = \frac{1 - x^2}{1 + x^2}.$$

Hence, the harmonic measure on the boundary is just given by $\delta_{x=1}$ for $x \in [0, \infty]$ or $\delta_{z=0}$ for $z \in [-1, 1]$.

Let us denote the N-th space by $H_{\eta(N)} = \ell^2\big((i, j), \tau_{\eta(N)}\big)$. We have the operator $\nabla_N : H_{\eta(N)} \to H_{\eta(N-1)}$ and its adjoint $\nabla_N^* : H_{\eta(N-1)} \to H_{\eta(N)}$, which are the limit $\alpha = \beta \to \infty$ of the operators for the η-β-chain;

$$\nabla_N \varphi(i,j) = \varphi(i,j+1) - \varphi(i+1,j)$$

Put $\nabla_N^+ := N\nabla_N^*$. Then it can be written as

$$\nabla_N^+ \varphi(i,j) = j\varphi(i,j-1) - i\varphi(i-1,j).$$

Then we have the ladder

$$H_{\eta(N)} \underset{\nabla_N^+}{\overset{\nabla_N}{\rightleftarrows}} H_{\eta(N-1)} \underset{\nabla_{N-1}^+}{\overset{\nabla_{N-1}}{\rightleftarrows}} H_{\eta(N-2)} \ .$$

This again satisfies the Heisenberg relation up the ladder

$$\nabla_N \nabla_N^+ - \nabla_{N-1}^+ \nabla_{N-1}^+ = \mathrm{id}_{H_N}.$$

We define the basis of H_N by

$$\tilde{\varphi}_{\eta(N),m} = (-1)^m \frac{1}{m!} (\nabla^+)^m \mathbf{1}_{H_{\eta(N-m)}}$$

(we normalize $\tilde{\varphi}$ soon.) It has an expression as the limit $\alpha \to \infty$ of the real Hahn function;

$$\begin{aligned} \tilde{\varphi}(i,j)_{\eta(N),m} &= 2^m \lim_{\alpha\to\infty} \varphi^{\alpha,\alpha}_{\eta(N),m}(i,j) = \sum_{0\le k\le m} \binom{i}{k}\binom{j}{m-k}(-1)^{m-k} \\ &= \sigma_{N,m}(\underbrace{1,\ldots,1}_{i}, \underbrace{-1,\ldots,-1}_{j}), \end{aligned}$$

where $\sigma_{N,m}$ is the m-th elementary symmetric function of N variable:

$$\sigma_{N,m}(x_1,\ldots,x_N) = \sum_{1\le j_1<\cdots<j_m\le N} x_{j_1}\cdots x_{j_m}.$$

We also have the following formulas by using the Heisenberg relation.

$$\begin{aligned} -\nabla_N^+ \tilde{\varphi}_{\eta(N-1),m-1} &= m\tilde{\varphi}_{\eta(N),m}, \\ -\nabla_N \tilde{\varphi}_{\eta(N),m} &= \tilde{\varphi}_{\eta(N-1),m-1}. \end{aligned}$$

Hence we have

$$\nabla_N^* \nabla_N \tilde{\varphi}_{\eta(N),m} = m\tilde{\varphi}_{\eta(N),m}$$

This shows that the function $\tilde{\varphi}_{N,m}$ is the eigen function of the number operator $\nabla_N^*\nabla_N$ with eigenvalue m, whence $\tilde{\varphi}_{\eta(N),m}$ is the orthogonal basis for the N-th layer. The L^2-norm is calculated as

$$||\tilde{\varphi}_{\eta(N),m}||^2_{H_{\eta(N)}} = \binom{N}{m}$$

To see the limit of these at the boundary, we need to scale. Let us scale the operator ∇_N as follows;

$$D_N := \sqrt{2\pi N}\nabla_N$$

Then the adjoint operator is given by

$$D_N^* = \sqrt{\frac{2\pi}{N}} \nabla_N^+.$$

Similarly we put

$$\varphi_{\eta(N),m} := (-1)^m \frac{1}{m!} (D^*)^m \mathbf{1}_{H_{\eta(N-m)}}$$

This is the old function up to a constant, that is,

$$\varphi_{\eta(N),m} = \frac{(2\pi)^{\frac{m}{2}}}{\big(N(N-1)\cdots(N-m+1)\big)^{\frac{1}{2}}} \tilde{\varphi}_{\eta(N),m}.$$

Hence this also gives an orthogonal basis and the L^2-norm of it is easily calculated as

$$||\varphi_{\eta(N),m}||^2_{H_{\eta(N)}} = \frac{(2\pi)^m}{m!}.$$

Now this is much better than the one for $\tilde{\varphi}_{\eta(N),m}$ because it is independent on N. Let us associate (i, j) with the point

$$w(i,j) := \frac{i-j}{\sqrt{2\pi(i+j)}} = \sqrt{\frac{N}{2\pi}} \frac{i-j}{i+j} \in \Big[-\sqrt{\frac{N}{2\pi}}, \sqrt{\frac{N}{2\pi}}\Big].$$

Then the measure τ_N becomes the measure

$$\tau_{\eta(N)} = \frac{1}{2^N} \sum_{i+j=N} \frac{N!}{i!j!} \delta_{\frac{i-j}{\sqrt{2\pi N}}} = \Big(\frac{1}{2}\delta_{\frac{1}{\sqrt{2\pi N}}} + \frac{1}{2}\delta_{-\frac{1}{\sqrt{2\pi N}}}\Big)^{*N}.$$

Here $*N$ denotes the N times additive-convolution. By the central limit theorem, this converges as $N \to \infty$;

$$\tau_{\eta(N)} \longrightarrow e^{-\pi w^2} dw$$

This is why we consider the scaled limit.

Now the limit $N \to \infty$ of the operators D_N and D_N^* converges as follows;

$$D_N \longrightarrow D = -\frac{\partial}{\partial w},$$
$$D_N^* \longrightarrow D^* = e^{\pi w^2} \frac{\partial}{\partial w} e^{-\pi w^2} = \frac{\partial}{\partial w} - 2\pi w.$$

These act on the space $H_\eta := L^2(\mathbb{R}, e^{-\pi w^2} dw)$. They satisfies the Heisenberg relation

$$DD^* - D^*D = (2\pi)\mathrm{id}_{H_\eta}.$$

And we obtain the basis

$$\varphi_{\eta,m} := (-1)^m \frac{1}{m!}(D^*)^m \mathbf{1}_{H_\eta}.$$

We have also an expression of the basis as as the limit of the finite layer;

$$\varphi_{\eta,m}(w) = \lim_{w(i,j)\to w} \varphi_{\eta(i+j),m}(i,j).$$

Also it can be written as the scaled limit $\alpha = \beta \to \infty$ of the real Jacobi basis;

$$\varphi_{\eta,m}(w) = \lim_{\alpha\to\infty} \varphi^{\alpha,\alpha}_{\eta(N),m}(z)\Big|_{z=\frac{2\pi}{\alpha}w} = e^{\pi w^2}\frac{(-1)^m}{m!}\Big(\frac{\partial}{\partial w}\Big)^m e^{-\pi m\alpha}$$

and these are the even and odd Hermite polynomials top-right of (Fig. 4.1).

Now we are looking at the parameters (α, β) of real β-chain. We first notice the special point $(1,1)$. At this point, we have a unique probability measure on the projective line that is invariant under the orthogonal group. Similarly, at the point $(2,2)$, we have a unique probability measure on the projective line over the complex which is invariant under the unitary group. Let us take the limit of one parameter, say α, as $\alpha \to \infty$. If $\beta = 1$ and $\alpha \to \infty$, we obtain the additive Haar measure $|x|^1_\eta d^*x = dx$ and the Laguerre basis for $L^2\big(\mathbb{R}/\{\pm 1\}, e^{-\pi x^2}|x|^1_\eta d^*x\big)$, which are essentially the even Hermite

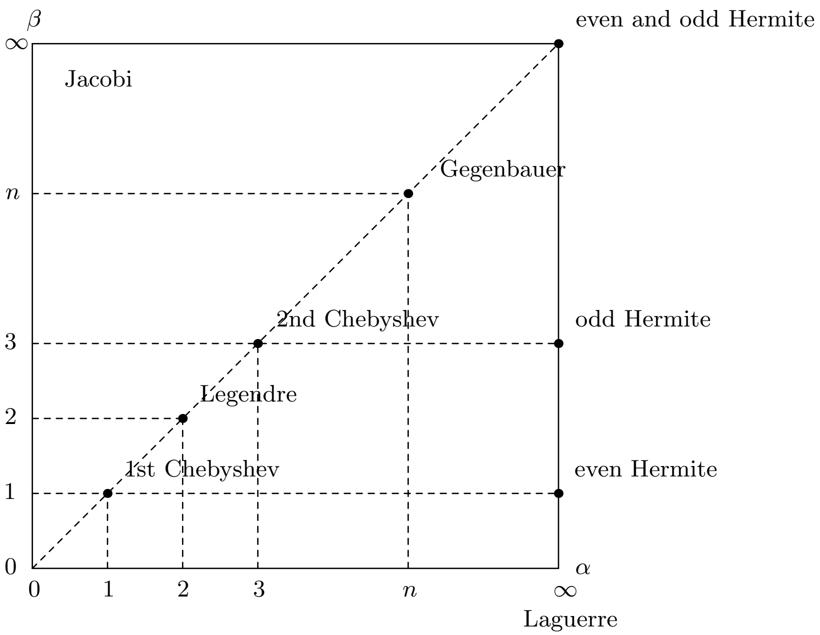


Fig. 4.1. η-β-chain (α, β)

polynomials. Remember the operators D and D^*. For the real case, they act between the space with $\beta = 1$ to $\beta = 3$. For $\beta = 3$, we have also the Hilbert space $L^2\big(\mathbb{R}/\{\pm 1\}, e^{-\pi x^2}|x|_\eta^3 d^*x\big)$, and the Laguerre basis is essentially the odd Hermite polynomials divided by x. Now if we take the limit $\alpha = \beta \to \infty$ diagonally, at the point (∞, ∞), we get the space $L^2\big(\mathbb{R}, e^{-\pi x^2} dw\big)$ and the basis for this space consisting of both odd and even Hermite polynomials.

5

q-Interpolation of Local Tate Thesis

Summary. In Sect. 5.1 we give the q-interpolation between the p-adic and the real theory of the local (and unramified) part of Tate's thesis. We study the Hilbert space $H^\beta = L^2(G, \tau^\beta)$ where

$$G = \begin{cases} \mathbb{Q}_p^*/\mathbb{Z}_p^* = p^{\mathbb{Z}} & \text{for } p, \\ \mathbb{R}^*/\{\pm 1\} = \mathbb{R}^+ & \text{for } \eta, \\ g^{\mathbb{Z}} & \text{for } q, \end{cases} \qquad \tau^\beta(x) = |x|^\beta \frac{d^*x}{\zeta(\beta)}.$$

Here the measure d^*x is the Haar measure (resp. counting measure) on G for η (resp. p and q) and $|\cdot|$ is the corresponding absolute values:

$$|a| = \begin{cases} p^{-\mathrm{ord}_p(a)} & \text{for } p, \\ \text{usual absolute value of } a & \text{for } \eta, \\ q^n \quad (a = g^n) & \text{for } q. \end{cases}$$

The measure τ^β is called the Tate measure. Note that the following stories are true for all subscripts p, η and q. Let $\pi^\beta : G \to U(H^\beta)$ be the unitary representation defined by

$$\pi^\beta(a)\varphi(x) := |a|^{-\frac{\beta}{2}} \varphi(a^{-1}x).$$

Let $\widehat{H}^\beta = L_2(\widehat{G}, \widehat{\tau}^\beta)$, where $\widehat{G}$ is the dual group of G;

$$\widehat{G} = \begin{cases} i\mathbb{R}/\frac{2\pi i}{\log p}\mathbb{Z} & \text{for } p, \\ i\mathbb{R} & \text{for } \eta, \\ i\mathbb{R}/\frac{2\pi i}{\log q}\mathbb{Z} & \text{for } q, \end{cases} \qquad \widehat{\tau}^\beta(s) = \zeta\Big(\frac{\beta}{2} + s, \frac{\beta}{2} - s\Big) d^\circ s,$$

and

$$d^\circ(it) = dt \cdot \begin{cases} \log p/2\pi & \text{for } p, \\ 1/4\pi & \text{for } \eta, \\ |\log q|/2\pi & \text{for } q. \end{cases}$$

We have the cyclic vector $\phi_{\mathbb{Z}}$ of H^β;

$$\phi_{\mathbb{Z}}(x) = \begin{cases} \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p, \\ 0 & \text{if } x \notin \mathbb{Z}_p, \end{cases} & \text{for } p, \\ e^{-\pi x^2} & \text{for } \eta, \\ 1/\zeta(1+n) \quad (x = g^n) & \text{for } q, \end{cases}$$

and the decomposition of the representation π^β into the continuous sum (= integral) of the 1-dimensional irreducible representations given by the characters in $\widehat{G}$ is as follows;

$$\begin{aligned} H^\beta &\xrightarrow{\sim} \widehat{H}^\beta \\ \phi_{\mathbb{Z}} &\longmapsto \mathbf{1}, \\ \pi^\beta(a)\phi_{\mathbb{Z}} &\longmapsto |a|^s \cdot \mathbf{1}. \end{aligned}$$

More generally, the isomorphism is given by

$$\varphi \longmapsto \tau^{\frac{\beta}{2}+s}(\varphi) = \int_G \varphi(x)\tau^{\frac{\beta}{2}+s}(dx)$$

In Sect. 5.2 we describe the Fourier–Bessel transform $\mathcal{F}^\beta$ on H^β. It is the operator that intertwines the representation $\pi^\beta(a)$ and $\pi^\beta(a^{-1})$: $\mathcal{F}^\beta\pi^\beta(a) = \pi^\beta(a^{-1})\mathcal{F}^\beta$, and it corresponds to the operator $\widehat{\mathcal{F}}^\beta$ on $\widehat{H}^\beta$ given by

$$\widehat{\mathcal{F}}^\beta : \widehat{H}^\beta \longrightarrow \widehat{H}^\beta; \qquad \widehat{f}(s) \longmapsto \widehat{f}(-s).$$

We have for any $\varphi \in H^\beta$

$$\tau^s(\mathcal{F}^\beta\varphi) = \tau^{\beta-s}(\varphi)$$

and this characterizes the Fourier–Bessel transform. Then we obtain the explicit expression of the kernel $\mathcal{F}^\beta$ (using the Bessel function for the real case). Notice that, in the q-world, the kernel $\frac{1}{\zeta_q(\beta)}\mathcal{F}_q^\beta(g^{n-1})$ is symmetric in the spectral parameter β and the geometric one n. This symmetry is lost in the p-adic and the real limits.

We have also the convolution and co-convolution structure;

$$H^\beta \otimes H^\beta \underset{\Delta_\beta}{\overset{*_\beta}{\rightleftarrows}} H^\beta$$

For the case of q, this interpolates the convolution and co-convolution structure on $\mathcal{S}(\mathbb{Q}_p^{\oplus n})^{GL_n(\mathbb{Z}_p)}$ for p, and on $\mathcal{S}(\mathbb{R}^{\oplus n})^{O_n}$ (resp. $\mathcal{S}(\mathbb{C}^{\oplus n})^{U_n}$) for the real (resp. complex) η. The operators $*_\beta$ and Δ_β are adjoint to each other. Remark that the convolution is not defined on the Hilbert space but the Schwartz space. We also obtain the following equivalent expressions:

$$\begin{aligned} \Delta_\beta\mathcal{F}^\beta(x,y) &= \mathcal{F}^\beta(x)\cdot\mathcal{F}^\beta(y), \\ \mathcal{F}^\beta(\varphi_1 *_\beta \varphi_2) &= \mathcal{F}^\beta(\varphi_1)\cdot\mathcal{F}^\beta(\varphi_2). \end{aligned}$$

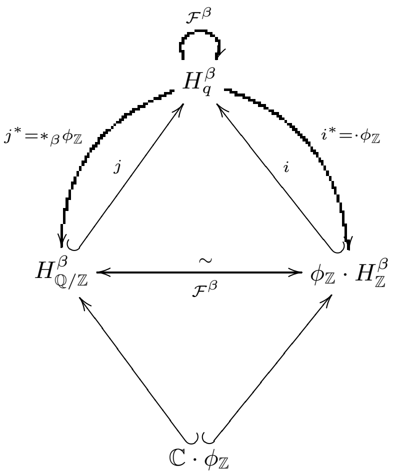
In Sect. 5.3 we introduce the Basic Basis, which does not have the spectral parameter

$$\phi_{m,q}(g^n) = \frac{\zeta_q(1)}{\zeta_q(1+m)} \frac{q^{m(m+n)}}{\zeta_q(1+m+n)}$$
$$\to \begin{cases} \phi_{m,p}(x) = \begin{cases} \phi_{\mathbb{Z}_p} & \text{if } m = 0, \\ \phi_{p^{-m}\mathbb{Z}_p^*} & \text{if } m > 0, \end{cases} & \text{for } ⓟ, \\ \phi_{m,\eta}(x) = \frac{(\pi x^2)^m}{m!} e^{-\pi x^2} & \text{for } ⓔ. \end{cases}$$

We denote by $H^\beta_{\mathbb{Q}/\mathbb{Z}}$ the Hilbert space generated by $\{\phi_m\}_{m\geq 0}$ with norm

$$||\phi_m||_{H^\beta_{\mathbb{Q}/\mathbb{Z}}} = C^\beta_q(m) = \frac{(1-q^\beta)\cdots(1-q^{\beta+m-1})}{(1-q)\cdots(1-q^m)} q^{-m\beta}$$
$$\to \begin{cases} \begin{cases} 1 & \text{if } m = 0, \\ (1-p^{-\beta})p^{m\beta} & \text{if } m > 0, \end{cases} & \text{for } ⓟ, \\ \dfrac{(\beta)_m}{m!} & \text{for } ⓔ. \end{cases}$$

The boundary space $H^\beta_{\mathbb{Z}} = L^2(G, \phi_{\mathbb{Z}}\tau^\beta)$ of the Markov chain after multiplication by $\phi_{\mathbb{Z}}$ is a subspace in H^β_q, for which we have the Laguerre basis $\phi_{\mathbb{Z}} \cdot \varphi^\beta_m$. $H^\beta_{\mathbb{Q}/\mathbb{Z}}$ is also a subspace in H^β_q, and the Fourier–Bessel transform $\mathcal{F}^\beta$ interchange $H^\beta_{\mathbb{Q}/\mathbb{Z}}$ and $\phi_{\mathbb{Z}} \cdot H^\beta_{\mathbb{Z}}$, carrying the basic basis to the Laguerre basis. Then we have the following diagram for q, p and η:



$$\phi_m \overset{\mathcal{F}^\beta}{\longleftrightarrow} \phi_{\mathbb{Z}} \cdot \varphi^\beta_{\mathbb{Z},m}$$

Here, for the p-adic case, the embedding i and j are the unitary embeddings and give "$\frac{1}{2}$-space" of H^β_p, which is the space of the function supported at $p^{\mathbb{N}} \subseteq p^{\mathbb{Z}}$. On the other hand for η and q cases, these are dense embeddings. The adjoint j^* is given by β-convolution with $\phi_{\mathbb{Z}}$.

We then introduce some invertible operators X and Y on H_q^β by

$$X\varphi(g^n) := q^n\varphi(g^n), \qquad Y\varphi(g^n) := \varphi(g^{n-1}).$$

They satisfy the q-commutativity relation;

$$XY = qYX.$$

Let $N := X + Y$, $A_+ := (1-q)^{-1}Y^{-1}X$ and $A_- := Y[1 - X^{-1}(1-Y)]$. These operators are nice in terms of the basic basis. Actually, we have

$$N\phi_m = q^{-m}\phi_m, \qquad A_+\phi_m = [m+1]_q q^{-m}\phi_{m+1}, \qquad A_-\phi_m = \phi_{m-1}.$$

They satisfy the $\mathfrak{sl}_2$-relations $A_\pm N = q^{\pm 1}NA_\pm$, $A_-A_+ - A_+A_- = N$, and an extra relation

$$A_-A_+ - q^{-1}A_+A_- = \mathrm{id}.$$

It is this extra relation that gives rise to the Heisenberg relations up the ladder.

5.1 Mellin Transforms

5.1.1 Classical Cases

Here we study the unitary representation $\pi : G \to U(H)$ of a group G. We assume that G is non-compact and commutative so that the representation theory is very easy. In this section, we mainly treat the case

$$G \simeq \begin{cases} \mathbb{Z} & \text{for } p \text{ or } q, \\ \mathbb{R} & \text{for } \eta. \end{cases}$$

One can also write these groups in a multiplicative way:

$$G_p := \mathbb{Q}_p^*/\mathbb{Z}_p^* = \begin{cases} p^{\mathbb{Z}} & \text{for } p, \\ \mathbb{R}^+ = e^{\mathbb{R}} & \text{for } \eta. \end{cases}$$

Here we denote by $\mathbb{R}^+$ the set of all positive real numbers. For the p-adic case, all the representation of G_p is determined by $\pi(p) \in U(H)$. On the other hand, for the real case, it can be written as $\pi(e^t) = e^{itA}$, where $A : H \to H$ is a self adjoint operator; A is the infinitesimal generator of the unitary group $\pi(e^t)$, it is defined by

$$iA\varphi = \frac{d}{dt}\bigg|_{t=0} \pi(e^t)\varphi.$$

We assume that the representations of these groups are multiplicity free, in the sense that there exists a cyclic vector $\phi \in H$. Namely, $\mathrm{Span}\{\pi(g)\phi \,\big|\, g \in G_p\}$ is dense in H. Let $\rho(g) := (\phi, \pi(g)\phi)_H$. Then $\rho : G_p \to \mathbb{C}$ is a positive definite function in the following sense: Define the inner product $\langle \cdot,\cdot\rangle_{C_c(G_p)} : C_c(G_p) \times C_c(G_p) \to \mathbb{C}$, where $C_c(G_p)$ is the set of continuous functions $f : G_p \to \mathbb{C}$ which have compact support, as follows:

$$\langle f_1, f_2\rangle_{C_c(G_p)} := \iint_{G_p\times G_p} dg_1 dg_2 f_1(g_1)\overline{f_2(g_2)}\rho(g_1^{-1}g_2) \qquad (f_1, f_2 \in C_c(G_p)).$$

Then this is a positive definite inner product on $C_c(G)$. Let us write

$$\pi(f)\phi := \int_{G_p} dg f(g)\pi(g)\phi \in H \qquad (f \in C_c(G_p)).$$

Then we have

$$\begin{aligned}\big(\pi(f_1)\phi, \pi(f_2)\phi\big)_H &= \iint_{G_p\times G_p} dg_1 dg_2 f_1(g_1)\overline{f_2(g_2)}(\pi(g_1)\phi, \pi(g_2)\phi)_H \\ &= \iint_{G_p\times G_p} dg_1 dg_2 f_1(g_1)\overline{f_2(g_2)}\rho(g_1^{-1}g_2) \\ &= \langle f_1, f_2\rangle_{C_c(G_p)}.\end{aligned}$$

This shows that H is the completion of $C_c(G_p)$ with respect to the inner product $\langle \cdot,\cdot\rangle_{C_c(G_p)}$.

To decompose the representation, we look at the dual group

$$\widehat{G}_p := \left(\mathbb{Q}_p^*/\mathbb{Z}_p^*\right)^\wedge = \begin{cases} i\mathbb{R}/\dfrac{2\pi i}{\log p}\mathbb{Z} & \text{for } p \neq \eta, \\ i\mathbb{R} & \text{for } p = \eta. \end{cases}$$

We denote an element of $\mathbb{Q}_p^*$ by a and an element of the dual $\widehat{\mathbb{Q}_p^*}$ of $\mathbb{Q}_p^*$ by s (i.e., s is a character of $\mathbb{Q}_p^*$). The duality $\langle\cdot,\cdot\rangle : G_p \times \widehat{G}_p \to \mathbb{C}$ is given by

$$\langle a, s\rangle = |a|_p^s \qquad (a \in G_p,\ s \in \widehat{G}_p).$$

For a function f on G_p, the Fourier transform $\widehat{f}$ (note that this is a function on $\widehat{G}_p$) is just given by the Mellin transform:

$$\widehat{f}(s) = \int_{G_p} f(a)|a|_p^s d^*a$$

and its inverse transform is given by

$$f(a) = \int_{\widehat{G}_p} \widehat{f}(s)|a|_p^{-s} d^\circ s,$$

where $d^\circ s$ is the measure on $\widehat{G}_p$ which is normalized by this duality:

$$d^\circ(it) = dt \cdot \begin{cases} \dfrac{\log p}{2\pi} & \text{for } p \neq \eta, \\ \dfrac{1}{4\pi} & \text{for } p = \eta. \end{cases}$$

The properties of the Mellin transform is as follows:

$$(f_1 * f_2)^\wedge(s) = \widehat{f}_1(s)\cdot\widehat{f}_2(s), \qquad \widehat{f^*}(s) = \overline{\widehat{f}(s)}$$

where $f^*(a) := \overline{f(a^{-1})}$.

Now let us decompose the representation. All we need to do is to introduce the positive function

$$\widehat{\rho}(s) := \int_{G_p} (\phi, \pi(a)\phi)_H |a|_p^s d^*a.$$

Then $\widehat{\rho}(s)d^\circ s$ is a measure on $\widehat{G}_p$. Since G_p is non-compact, H decomposes as the continuous sum of the irreducible one dimensional representation of G_p. This decomposition is given by

$$\begin{aligned} H &\xrightarrow{\sim} L^2\big(\widehat{G}_p, \widehat{\rho}(s)d^\circ(s)\big) =: \widehat{H} \\ \phi &\longmapsto \mathbf{1}, \\ \pi(a)\phi &\longmapsto |a|_p^s\cdot\mathbf{1} \qquad (a\in G_p), \\ \pi(f)\phi &\longmapsto \widehat{f}(s)\cdot\mathbf{1} \qquad (f\in C_c(G_p)). \end{aligned}$$

Since the set of element $\pi(f)\phi$ is dense in H, to check that the above is a well-defined isometry, it is enough to note that

$$\big(\pi(f_1)\phi, \pi(f_2)\phi\big)_H = f_1 * f_2^* * \rho(1) = \int_{\widehat{G}_p} \widehat{f}_1(s)\overline{\widehat{f}_2(s)}\widehat{\rho}(s)d^\circ s.$$

In the case of the reals, we also have that the operator iA on H corresponds under the above isomorphism to the operator of multiplication by s on $\widehat{H}$.

We specialize this general theory to the context of Tate thesis [Ta]. In Tate thesis, we consider the unitary representation $\pi : \mathbb{Q}_p^* \to U\big(L^2(\mathbb{Q}_p, dx)\big)$ defined by

$$\pi(a)\varphi(x) := |a|_p^{-1/2}\varphi(a^{-1}x) \qquad (\varphi\in L^2(\mathbb{Q}_p, dx)).$$

We always specialize to the “unramified” part, which means that we are taking $\mathbb{Z}_p^*$-invariants, and consider only

$$\pi : \mathbb{Q}_p^*/\mathbb{Z}_p^* \longrightarrow U(L^2(\mathbb{Q}_p, dx)^{\mathbb{Z}_p^*}).$$

The unramified part was also studied by Iwasawa independently of Tate. Let us generalize the above as follows. The Tate measure $\tau_p^\beta(x)$ on $\mathbb{Q}_p$ is given for $\beta > 0$ by

$$\tau_p^\beta(x) := |x|_p^\beta \frac{d^*x}{\zeta_p(\beta)}.$$

Note that $\tau_{\mathbb{Z}_p}^{\beta}(x) = \phi_{\mathbb{Z}_p}(x) \cdot \tau_p^{\beta}(x)$ is the γ-measure which we have studied before with

$$\phi_{\mathbb{Z}_p}(x) = \begin{cases} \text{the characteristic function on } \mathbb{Z}_p & \text{for } p \neq \eta, \\ e^{-\pi|x|_\eta^2} & \text{for } p = \eta. \end{cases}$$

Let

$$H_p^{\beta} := L^2(\mathbb{Q}_p, \tau_p^{\beta})^{\mathbb{Z}_p^*}$$

For example if $\beta = 1$, $\tau_p^1(x) = dx$ is the additive Haar measure on $\mathbb{Q}_p$. Further if $\beta = n$ for some positive integer n, H_p^n can be identified with

$$H_p^n \simeq L^2(\mathbb{Q}_p^{\oplus n}, dx)^{GL_n(\mathbb{Z}_p)}.$$

Any element $\varphi \in L^2(\mathbb{Q}_p^{\oplus n}, dx)^{GL_n(\mathbb{Z}_p)}$ that is invariant under the compact group $GL_n(\mathbb{Z}_p)$ can be written as $\varphi(x_1, \ldots, x_n) = \widehat{\varphi}(|x_1, \ldots, x_n|_p)$ with $\widehat{\varphi} \in H_p^n$. Let $\pi_p^{\beta} : G_p \to U(H_p^{\beta})$ be the unitary representation defined by

$$\pi_p^{\beta}(a)\varphi(x) := |a|_p^{-\beta/2}\varphi(a^{-1}x)$$

We next consider the decomposition of this representation. We can take the cyclic vector of H_p^{β} as $\phi_{\mathbb{Z}_p}$. We get the positive definite function

$$\rho_p^{\beta}(a) := \left(\phi_{\mathbb{Z}_p}, \pi_p^{\beta}(a)\phi_{\mathbb{Z}_p}\right)_{H_p^{\beta}} = |a|_p^{-\beta/2} \int_{\mathbb{Q}_p} \phi_{\mathbb{Z}_p}(x)\phi_{\mathbb{Z}_p}(a^{-1}x)|x|_p^{\beta} \frac{d^*x}{\zeta_p(\beta)}$$

Then it is shown that

$$\rho_p^{\beta}(a) = \rho_{\infty}(a)^{\frac{\beta}{2}}\rho_0(a)^{\frac{\beta}{2}}. \tag{5.1}$$

In fact, for the case of $p \neq \eta$, we have

$$\begin{aligned} \rho_p^{\beta}(p^n) &= p^{n\frac{\beta}{2}} \int_{\mathbb{Q}_p} \phi_{\mathbb{Z}_p}(x)\phi_{p^n\mathbb{Z}_p}(x)|x|_p^{\beta} \frac{d^*x}{\zeta_p(\beta)} \\ &= \begin{cases} p^{n\frac{\beta}{2}} \displaystyle\int_{\mathbb{Q}_p} \phi_{p^n\mathbb{Z}_p}(x)|x|_p^{\beta} \frac{d^*x}{\zeta_p(\beta)} = p^{-n\frac{\beta}{2}} & \text{if } n \geq 0, \\ p^{n\frac{\beta}{2}} \displaystyle\int_{\mathbb{Q}_p} \phi_{\mathbb{Z}_p}(x)|x|_p^{\beta} \frac{d^*x}{\zeta_p(\beta)} = p^{n\frac{\beta}{2}} & \text{if } n < 0 \end{cases} \\ &= |1, p^n|_p^{-\frac{\beta}{2}}|1, p^{-n}|_p^{-\frac{\beta}{2}}. \end{aligned}$$

On the other hand for the case of $p = \eta$, we have

$$\begin{aligned} \rho_{\eta}^{\beta}(a) &= |a|_{\eta}^{-\frac{\beta}{2}} \int_{\mathbb{R}} e^{-\pi|x|_{\eta}^2(1+|a|_{\eta}^{-2})}|x|_{\eta}^{\beta} \frac{d^*x}{\zeta_{\eta}(\beta)} \\ &= |a|_{\eta}^{-\frac{\beta}{2}}(1 + |a|_{\eta}^{-2})^{-\frac{\beta}{2}} \\ &= \left(1 + |a|_{\eta}^2\right)^{-\frac{\beta}{4}}\left(1 + |a|_{\eta}^{-2}\right)^{-\frac{\beta}{4}} \\ &= |1, a|_{\eta}^{-\frac{\beta}{2}}|1, a^{-1}|_{\eta}^{-\frac{\beta}{2}}. \end{aligned}$$

This shows the claim (5.1).

Let us calculate the dual measure $\widehat{\tau}_p^\beta$ of τ_p^β. By (5.1) and the formula $|a|_p = \rho_0(a)/\rho_\infty(a)$ we have

$$\begin{aligned}\widehat{\tau}_p^\beta(s) := \widehat{\rho}_p^\beta(s) d^\circ s &= d^\circ s \int_{\mathbb{Q}_p^*} \rho_p^\beta(a)|a|_p^s d^* a \\ &= d^\circ s \int_{\mathbb{Q}_p^*} \rho_\infty(a)^{\frac{\beta}{2}-s} \rho_0(a)^{\frac{\beta}{2}+s} d^* a \\ &= d^\circ s \cdot \zeta_p\Big(\frac{\beta}{2}+s, \frac{\beta}{2}-s\Big) \\ &= \Big|\zeta_p\Big(\frac{\beta}{2}+s\Big)\Big|_p^2 \frac{d^\circ s}{\zeta_p(\beta)}.\end{aligned}$$

Hence we have the decomposition

$$\begin{aligned}H_p^\beta = L^2(\mathbb{Q}_p, \tau_p^\beta)^{\mathbb{Z}_p^*} &\xrightarrow{\sim} L^2\big(\widehat{G}_p, \widehat{\tau}_p^\beta\big) =: \widehat{H}_p^\beta \\ \phi_{\mathbb{Z}_p} &\longmapsto \mathbf{1}, \\ \pi_p^\beta(a)\phi_{\mathbb{Z}_p} &\longmapsto |a|_p^s \cdot \mathbf{1} \qquad (a \in G_p).\end{aligned}$$

The isomorphism is given by

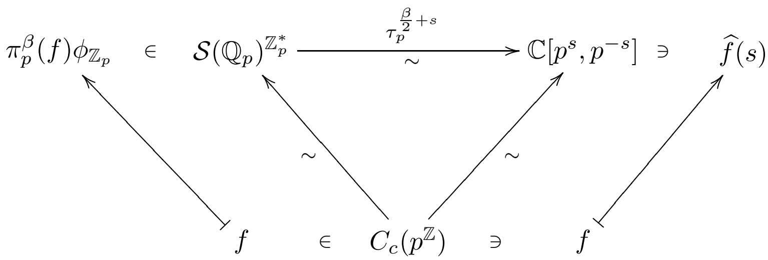
$$\varphi \longmapsto \tau_p^{\frac{\beta}{2}+s}(\varphi) = \int_{\mathbb{Q}_p} \varphi(a)|a|^{\frac{\beta}{2}+s} \frac{d^* a}{\zeta_p\big(\frac{\beta}{2}+s\big)}.$$

In the case of $p = \eta$, we have also the infinitesimal generator

$$\begin{aligned}iA\varphi(x) = \pi_\eta^\beta\Big(a\frac{\partial}{\partial a}\Big|_{a=1}\Big)\varphi(x) &= \frac{\partial}{\partial t}\Big|_{t=0} \pi_\eta^\beta(e^t)\varphi(x) \\ &= \frac{\partial}{\partial t}\Big|_{t=0} e^{-t\frac{\beta}{2}}\varphi(e^{-t}x) \\ &= -\Big(\frac{\beta}{2} + x\frac{d}{dx}\Big)\varphi(x)\end{aligned}$$

Namely, $i\big(\frac{\beta}{2} + x\frac{d}{dx}\big)$ is the self-adjoint infinitesimal generator.

One can obtain more precise information from the following commutative diagrams: In the case $p \neq \eta$, we have



where $\mathcal{S}(\mathbb{Q}_p)^{\mathbb{Z}_p^*}$ is the set of locally constant compactly supported functions on $\mathbb{Q}_p$ which are invariant under the action of $\mathbb{Z}_p^*$. For example, by the inverse map of $\tau_p^{\frac{\beta}{2}+s}$, the polynomial $F(p^{-s}, p^{-s})$ is mapped to $F(\pi_p^{\beta}(p), \pi_p^{\beta}(p^{-1}))\phi_{\mathbb{Z}_p}$. For the real case, we have

$$\begin{array}{ccc}
\mathbb{C}\big[-\big(\frac{\beta}{2} + x\frac{d}{dx}\big)\big]e^{-\pi x^2} \equiv \mathbb{C}[x^2]e^{-\pi x^2} & \xrightarrow[\sim]{\tau_\eta^{\frac{\beta}{2}+s}} & \mathbb{C}[s] \\
\nwarrow \sim & & \nearrow \sim \text{ (Mellin transform)} \\
& \mathbb{C}\big[a\frac{\partial}{\partial a}\big|_{a=1}\big] &
\end{array}$$

The inverse map of $\tau_\eta^{\frac{\beta}{2}+s}$ take the polynomial $F(s)$ to $F(\pi_q^B(a\frac{\partial}{\partial a}|_{a=1}))\phi_{\mathbb{Z}_\eta}$.

5.1.2 *q*-Interpolations

Now we want the q-interpolation of these local stories. We work on

$$G_q := g^{\mathbb{Z}}$$

for some letter g (we use the letter g different from q since in the p-adic limit we want to let $q \to 0$ but keep $g^n = p^n\mathbb{Z}_p^*$). We define the q-Tate measure for G_q by

$$\tau_q^{\beta}(g^n) := \frac{\zeta_q(1)}{\zeta_q(\beta)} q^{n\beta}.$$

This is a positive measure for $\beta > 0$, whence we have the Hilbert space $H_q^{\beta} := \ell^2(G_q, \tau_q^{\beta})$. The "$q$-absolute value" is defined on G_q by $|g^n|_q = q^n$. Let $\pi_q^{\beta} : G_q \to U(H_q^{\beta})$ be the unitary representation of G_q defined by

$$\pi_q^{\beta}(g^n)\varphi(g^m) := q^{-n\frac{\beta}{2}}\varphi(g^{m-n}).$$

For the cyclic vector, we take

Definition: $$\phi_{\mathbb{Z}_q}(g^n) := \frac{1}{\zeta_q(1+n)} = \begin{cases} \prod_{m>n}(1-q^m) & \text{for } n \geq 0, \\ 0 & \text{for } n < 0. \end{cases}$$

Then we have

$$\phi_{\mathbb{Z}_q} \to \begin{cases} \phi_{\mathbb{Z}_p} & \text{in } \textcircled{p} \text{ limit,} \\ \phi_{\mathbb{Z}_\eta} & \text{in } \textcircled{\eta} \text{ limit.} \end{cases}$$

Actually, let us calculate the p-adic and real limit of $\phi_{\mathbb{Z}_q}$. First of all, notice that

$$\lim_{q\to 0} \phi_{\mathbb{Z}_q}(g^n) = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n < 0. \end{cases}$$

Hence, in the p-adic limit ⓟ (recall that ⓟ means that $q \to 0$ in such a way that $q^\beta \to p^{-\beta}$ and $g^n \to p^n \mathbb{Z}_p^*$), we see that $\phi_{\mathbb{Z}_q} \to \phi_{\mathbb{Z}_p}$. This case is very simple because the group is the same; $G_q \simeq G_p$. On the other hand in the real limit ⓝ (recall also that ⓝ means that $q \to 1$, $n \to \infty$ in such a way that $\frac{q^n}{1-q} \to \pi x^2 \in \mathbb{R}/\{\pm 1\}$), since

$$\begin{aligned} \phi_{\mathbb{Z}_q}(g^n) &= \frac{1}{\zeta_q(1+n)} \\ &= \sum_{k\geq 0} \frac{\zeta_q(1)}{\zeta_q(1+k)} (-1)^k q^{\frac{k(k-1)}{2}} q^{k(1+n)} \\ &= \sum_{k\geq 0} \frac{(1-q)^k}{(1-q)(1-q^2)\cdots(1-q^k)} (-1)^k q^{\frac{k(k+1)}{2}} \left(\frac{q^n}{1-q}\right)^k \\ &\to \sum_{k\geq 0} \frac{1}{k!} (-1)^k (\pi x^2)^k \\ &= e^{-\pi x^2} = \phi_{\mathbb{Z}_\eta}(x), \end{aligned}$$

one gets that $\phi_{\mathbb{Z}_q} \to \phi_{\mathbb{Z}_\eta}$.

With the representation π_q^β and the cyclic vector $\phi_{\mathbb{Z}_q}$, there is associated the positive definite function on G_q

$$\begin{aligned} \rho_q^\beta(g^n) &:= (\phi_{\mathbb{Z}_q}, \pi_q^\beta(g^n)\phi_{\mathbb{Z}_q})_{H_q^\beta} \\ &= \sum_{k\geq 0} \phi_{\mathbb{Z}_q}(g^k) q^{-n\frac{\beta}{2}} \phi_{\mathbb{Z}_q}(g^{k-n}) \frac{\zeta_q(1)}{\zeta_q(\beta)} q^{k\beta}. \end{aligned} \tag{5.2}$$

We next calculate the Mellin transforms of $\rho(g^n)$. We denote the dual group of G_q by

$$\widehat{G}_q := i\mathbb{R}/\frac{2\pi i}{\log q}\mathbb{Z}$$

The q-Mellin transform of $f : G_q \to \mathbb{C}$ is given by

$$\widehat{f}_q(s) := \sum_n f(g^n) q^{ns} = \int_{G_q} f(x) |x|_q^s d^* x$$

Then the inverse transform is given by

$$f(g^n) = \int_{\widehat{G}_q} \widehat{f}_q(s) q^{-ns} d_q^\circ s, \qquad d_q^\circ(it) := dt \cdot \left|\frac{\log q}{2\pi}\right|.$$

From (5.2), we see that the q-Mellin transform $\widehat{\rho}_q^\beta$ of ρ_q^β is given by

$$\begin{aligned}
\widehat{\rho}_q^\beta(s) &= \sum_n \rho_q^\beta(g^n) q^{ns} \\
&= \frac{\zeta_q(1)}{\zeta_q(\beta)} \sum_{n,k\geq 0} \phi_{\mathbb{Z}_q}(g^k)\phi_{\mathbb{Z}_q}(g^{k-n}) q^{-n\frac{\beta}{2}+k\beta+ns} \\
&= \frac{\zeta_q(1)}{\zeta_q(\beta)} \sum_{m,k} \phi_{\mathbb{Z}_q}(g^k)\phi_{\mathbb{Z}_q}(g^m) q^{k\left(\frac{\beta}{2}+s\right)} q^{m\left(\frac{\beta}{2}-s\right)} \qquad (m := k-n) \\
&= \frac{1}{\zeta_q(1)\zeta_q(\beta)} \sum_{m,k\geq 0} \frac{\zeta_q(1)}{\zeta_q(1+k)} q^{k\left(\frac{\beta}{2}+s\right)} \frac{\zeta_q(1)}{\zeta_q(1+m)} q^{m\left(\frac{\beta}{2}-s\right)} \\
&= \frac{\zeta_q(\frac{\beta}{2}+s)\zeta_q(\frac{\beta}{2}-s)}{\zeta_q(1)\zeta_q(\beta)} \\
&= \zeta_q\Big(\frac{\beta}{2}+s, \frac{\beta}{2}-s\Big),
\end{aligned}$$

whence the dual measure is given by

$$\widehat{\tau}_q^\beta(s) := \widehat{\rho}_q^\beta(s) d^\circ s = \zeta_q\Big(\frac{\beta}{2}+s, \frac{\beta}{2}-s\Big) d^\circ s.$$

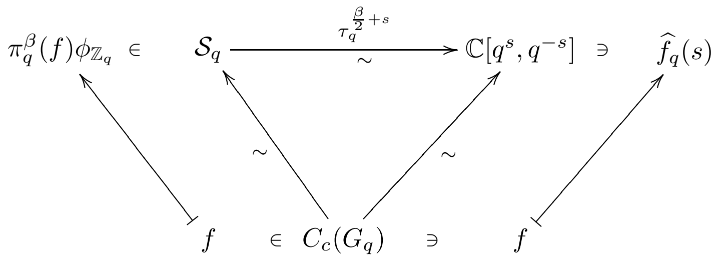
Therefore we obtain the decomposition of the representation H_q^β:

$$\begin{aligned}
H_q^\beta = \ell^2(G_q, \tau_q^\beta) &\xrightarrow{\sim} L_2\big(\widehat{G}_q, \widehat{\tau}_q^\beta\big) =: \widehat{H}_q^\beta \\
\phi_{\mathbb{Z}_q} &\longmapsto \mathbf{1}, \\
\pi_q^\beta(g^n)\phi_{\mathbb{Z}_q} &\longmapsto q^{ns} \cdot \mathbf{1}, \\
\pi_q^\beta(f)\phi_{\mathbb{Z}_q} &\longmapsto \widehat{f}_q(s) \cdot \mathbf{1} \qquad (f \in C_c(G_q)).
\end{aligned}$$

The isomorphism is given by

$$\varphi \longmapsto \tau_q^{\frac{\beta}{2}+s}(\varphi) = \frac{\zeta_q(1)}{\zeta_q\big(\frac{\beta}{2}+s\big)} \sum_n \varphi(g^n) q^{n\left(\frac{\beta}{2}+s\right)}.$$

Again one obtains more precise information from the following commutative diagram:



Here $\mathcal{S}_q := \mathrm{Span}\big\{\pi_q^\beta(g^n)\phi_{\mathbb{Z}_q}\big\}_{n\in\mathbb{Z}}$.

5.2 Fourier–Bessel Transforms

5.2.1 Fourier Transform on H_p^β

Recall that the Fourier transform $\mathcal{F} = \mathcal{F}_p^1$ is a unitary operator from the space $H_p = H_p^1 = L^2(\mathbb{Q}_p, dx)^{\mathbb{Z}_p^*}$ to itself. Moreover, $\mathcal{F}$ satisfies the following properties:

$$\begin{aligned} \mathcal{F} &= \mathcal{F}^* = \mathcal{F}^{-1}, \\ \mathcal{F}\phi_{\mathbb{Z}_p} &= \phi_{\mathbb{Z}_p}, \\ \mathcal{F}\pi_p^1(a) &= \pi_p^1(a^{-1})\mathcal{F}. \end{aligned}$$

Note that the third property shows that $\mathcal{F}$ intertwines the representations $\pi_p^1(a)$ and $\pi_p^1(a^{-1})$. Similarly if we consider the space $H_p^n = L^2(\mathbb{Q}_p^{\oplus n}, dx)^{GL_n(\mathbb{Z}_p)}$, we have the n-dimensional Fourier transforms $\mathcal{F}_p^n$. This is also a unitary operator from H_p^n to itself and satisfies

$$\begin{aligned} \mathcal{F}_p^n &= \left(\mathcal{F}_p^n\right)^* = \left(\mathcal{F}_p^n\right)^{-1}, \\ \mathcal{F}_p^n \phi_{\mathbb{Z}_p^{\oplus n}} &= \phi_{\mathbb{Z}_p^{\oplus n}}, \\ \mathcal{F}_p^n \pi_p^n(a) &= \pi_p^n(a^{-1})\mathcal{F}_p^n. \end{aligned}$$

More generally, for any $\beta > 0$, we have the operator $\mathcal{F}_p^\beta : H_p^\beta \to H_p^\beta$ satisfying

$$\begin{aligned} \mathcal{F}_p^\beta &= \left(\mathcal{F}_p^\beta\right)^* = \left(\mathcal{F}_p^\beta\right)^{-1}, \\ \mathcal{F}_p^\beta \phi &= \phi, \\ \mathcal{F}_p^\beta \pi_p^\beta(a) &= \pi_p^\beta(a^{-1})\mathcal{F}_p^\beta. \end{aligned}$$

The operator $\mathcal{F}_p^\beta$ on H_p^β corresponds to the operator $\widehat{\mathcal{F}}_p^\beta$ on $\widehat{H}_p^\beta$,

$$\tau_p^{\frac{\beta}{2}+s} : \qquad \underset{\circlearrowright\, \mathcal{F}_p^\beta}{H_p^\beta} \xrightarrow{\ \sim\ } \underset{\circlearrowright\, \widehat{\mathcal{F}}_p^\beta}{\widehat{\mathcal{F}}_p^\beta}$$

where the map $\widehat{\mathcal{F}}_p^\beta : \widehat{H}_p^\beta \to \widehat{H}_p^\beta$ is given by

$$\widehat{\mathcal{F}}_p^\beta \widehat{f}(s) := \widehat{f}(-s).$$

It is also clear that

$$\begin{aligned} \widehat{\mathcal{F}}_p^\beta &= \left(\widehat{\mathcal{F}}_p^\beta\right)^* = \left(\widehat{\mathcal{F}}_p^\beta\right)^{-1}, \\ \widehat{\mathcal{F}}_p^\beta \mathbf{1} &= \mathbf{1} \end{aligned}$$

because the measure $\widehat{\tau}_p^\beta(s)$ is symmetric in $s \mapsto -s$.

We can always write $\mathcal{F}_p^\beta$ by using a kernel $\mathcal{F}_p^\beta : G_p \times G_p \to \mathbb{C}$;

$$\mathcal{F}_p^\beta \varphi(y) = \int_{G_p} \varphi(x) \mathcal{F}_p^\beta(x, y) \tau_p^\beta(dx).$$

Since $\mathcal{F}_p^\beta$ intertwines the action of $\pi_p^\beta(a)$ and $\pi_p^\beta(a^{-1})$ for $a \in G_p$, we see that the kernel $\mathcal{F}_p^\beta(x,y)$ depend only on $x \cdot y$. Then it is easy to calculate that

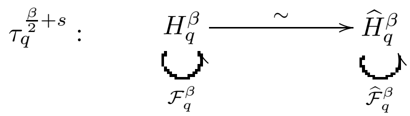
$$\mathcal{F}_p^\beta(x) = \zeta_p(\beta)\big(\phi_{\mathbb{Z}_p}(x) - p^{-\beta}\phi_{\mathbb{Z}_p}(px)\big),$$
$$\mathcal{F}_\eta^\beta(x) = \sum_{k\geq 0}(-1)^k \frac{\zeta_\eta(\beta)}{\zeta_\eta(\beta+2k)}\frac{1}{k!}(\pi x^2)^k = \zeta_\eta(\beta)|x|_\eta^{1-\frac{\beta}{2}}\pi J_{\frac{\beta}{2}-1}(2\pi x),$$

where $J_\beta(x)$ is the Bessel function. In particular for $\beta = 1$, we have

$$\mathcal{F}_p^1(x) = \frac{1}{1-p^{-1}}\big(\phi_{\mathbb{Z}_p}(x) - p^{-1}\phi_{\mathbb{Z}_p}(px)\big) = \phi_{\mathbb{Z}_p}(x) - \frac{1}{p-1}\phi_{p^{-1}\mathbb{Z}_p^*}(x),$$
$$\mathcal{F}_\eta^1(x) = \sum_{k\geq 0}\frac{(-1)^k(2\pi x^2)^k}{2k!} = \cos(2\pi x).$$

5.2.2 *q*-Fourier Transform

We now construct the counterpart q-theory of the previous subsection, that is, the q-Fourier transforms: The operator $\mathcal{F}_q^\beta$ on H_q^β is defined to be the operator that corresponds to $\widehat{\mathcal{F}}_q^\beta$ on $\widehat{H}_q^\beta$



where $\widehat{\mathcal{F}}_q^\beta \widehat{f}(s) = \widehat{f}(-s)$. Note that, since $\widehat{\tau}_q^\beta$ and $\mathbf{1}$ are invariant under $s \mapsto -s$, we have

$$\mathcal{F}_q^\beta = \big(\mathcal{F}_q^\beta\big)^* = \big(\mathcal{F}_q^\beta\big)^{-1},$$
$$\mathcal{F}_q^\beta\phi_{\mathbb{Z}_q} = \phi_{\mathbb{Z}_q},$$
$$\mathcal{F}_q^\beta\pi_q^\beta(g^n) = \pi_q^\beta(g^{-n})\mathcal{F}_q^\beta.$$

Similarly, $\mathcal{F}_q^\beta$ is given by the kernel $\mathcal{F}_q^\beta : G_q \to \mathbb{C}$:

$$\mathcal{F}_q^\beta\varphi(y) = \int_{G_q}\varphi(x)\mathcal{F}_q^\beta(x\cdot y)\tau_q^\beta(dx).$$

Then what is the kernel $\mathcal{F}_q^\beta(x)$? Note that $\mathcal{F}_q^\beta$ is defined by the equation

$$\tau_q^{\frac{\beta}{2}+s}(\mathcal{F}_q^\beta\varphi) = \tau_q^{\frac{\beta}{2}-s}(\varphi),$$

equivalently

$$\tau_q^s(\mathcal{F}_q^\beta\varphi) = \tau_q^{\beta-s}(\varphi), \tag{5.3}$$

or, explicitly,

$$\iint_{G_q\times G_q}\varphi(x)\mathcal{F}_q^\beta(x\cdot y)\tau_q^\beta(dx)\tau_q^s(dy)=\int_{G_q}\varphi(x)\tau_q^{\beta-s}(dx).$$

This is equivalent to the equation (for all φ's):

$$\int_{G_q}\varphi(x)\frac{\zeta_q(1)}{\zeta_q(\beta)}|x|_q^{\beta-s}d^*x\int_{G_q}\mathcal{F}_q^\beta(y)\frac{\zeta_q(1)}{\zeta_q(s)}|y|_q^s d^*y=\int_{G_q}\frac{\zeta_q(1)}{\zeta_q(\beta-s)}|x|_q^{\beta-s}\varphi(x)d^*x.$$

Notice that the integral is just the sum on $g^{\mathbb{Z}}$ and that $|g^n|_q=q^n$. Hence the kernel $\mathcal{F}_q^\beta(y)$ is uniquely characterized by (for $\mathrm{Re}(s)=\frac{\beta}{2}$),

$$\int_{G_q}\mathcal{F}_q^\beta(y)\frac{\zeta_q(1)}{\zeta_q(s)}|y|_q^s d^*y=\frac{\zeta_q(\beta)}{\zeta_q(\beta-s)}. \tag{5.4}$$

Actually, we claim that

$$\mathcal{F}_q^\beta(g^n)={}_1\phi_1(\infty;\beta;1+n)$$

$$(C_p)\qquad =\frac{\zeta_q(\beta)}{\zeta_q(1)}\sum_{k\geq 0}(-1)^k q^{\frac{k(k-1)}{2}}\frac{\zeta_q(1)}{\zeta_q(1+k)}\frac{\zeta_q(1)}{\zeta_q(1+n+k)}q^{\beta k}$$

$$(C_\eta)\qquad =\frac{\zeta_q(\beta)}{\zeta_q(1)}\sum_{k\geq 0}(-1)^k q^{\frac{k(k-1)}{2}}\frac{\zeta_q(1)}{\zeta_q(1+k)}\frac{\zeta_q(1)}{\zeta_q(\beta+k)}q^{(1+n)k}.$$

Then, for $0<\mathrm{Re}(s)<\beta$, we have

$$\begin{aligned}
&\int_{G_q}\mathcal{F}_q^\beta(y)\frac{\zeta_q(1)}{\zeta_q(s)}|y|_q^s d^*y\\
&\quad=\frac{\zeta_q(\beta)}{\zeta_q(s)}\sum_n q^{ns}\sum_{k\geq 0}(-1)^k q^{\frac{k(k-1)}{2}}\frac{\zeta_q(1)}{\zeta_q(1+k)}\frac{\zeta_q(1)}{\zeta_q(1+n+k)}q^{\beta k}\\
&\quad=\frac{\zeta_q(\beta)}{\zeta_q(s)}\sum_{m\geq 0}q^{ms}\frac{\zeta_q(1)}{\zeta_q(1+m)}\sum_{k\geq 0}(-1)^k q^{\frac{k(k-1)}{2}}\frac{\zeta_q(1)}{\zeta_q(1+k)}q^{(\beta-s)k}\quad (m:=n+k)\\
&\quad=\frac{\zeta_q(\beta)}{\zeta_q(s)}\zeta_q(s)\frac{1}{\zeta_q(\beta-s)}\\
&\quad=\frac{\zeta_q(\beta)}{\zeta_q(\beta-s)}.
\end{aligned}$$

This shows that $\mathcal{F}_q^\beta(g^n)$ as given by (C_p) satisfies the formula (5.4). Further, it is easy to see that the two expressions (C_p) and (C_η) are the same because expanding $1/\zeta_q(1+n+k)$ in (C_p) gives the expression

$$\frac{1}{\zeta_q(\beta)}\mathcal{F}_q^\beta(g^{n-1})=\sum_{k,\ell\geq 0}q^{k\ell}\left[(-1)^k q^{\frac{k(k-1)}{2}}\frac{\zeta_q(1)}{\zeta_q(1+k)}q^{\beta k}\right]\left[(-1)^\ell q^{\frac{\ell(\ell-1)}{2}}\frac{\zeta_q(1)}{\zeta_q(1+\ell)}q^{nk}\right],$$

which is symmetric in β and n. This symmetry between the spectral parameter β and the geometric parameter n is unique to the q-theory, and is lost in the p-adic and real limits. The expression for the kernel $\mathcal{F}_p^\beta$ (resp. $\mathcal{F}_\eta^\beta$) is obtained by taking the p-adic (resp. real) limits of the formula (C_p) (resp. (C_η)).

5.2.3 Convolutions

We here study the convolution structure. Let $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{Q}_p^{\oplus n})^{GL_n(\mathbb{Z}_p)}$. Then, we have the additive convolution of φ_1 and φ_2:

$$\varphi_1 *_n \varphi_2(y) := \int_{\mathbb{Q}_p^{\oplus n}} \varphi_1(x)\varphi_2(x-y)dx.$$

It is clear that $\varphi_1 *_n \varphi_2 \in C_c^\infty(\mathbb{Q}_p^{\oplus n})^{GL_n(\mathbb{Z}_p)}$ and the convolution product $*_n$ is associative and commutative. Further, it holds that

$$\mathcal{F}_p^n(\varphi_1 *_n \varphi_2) = \mathcal{F}_p^n(\varphi_1) \cdot \mathcal{F}_p^n(\varphi_2)$$

for the p-adic and real case. Next we want to describe the convolution structure which interpolates the convolution of the p-adic and real one. Define

$$\delta_q^\beta(x_1, \ldots, x_n) := \int_{G_q} \mathcal{F}_q^\beta(x_1 \cdot y) \cdots \mathcal{F}_q^\beta(x_n \cdot y)\tau_q^\beta(dy).$$

The function $\delta_q^\beta(x_1, \ldots, x_n)$ is well-defined, is symmetric in the variables $x_1, \ldots, x_n$ and homogeneous in the sense that

$$\delta_q^\beta(yx_1, \ldots, yx_n) = \delta_q^\beta(x_1, \ldots, x_n)|y|_q^{-\beta}.$$

For example, if we take $n = 1$

$$\delta_q^\beta(x) = |x|_q^{-\beta}\frac{\zeta_q(\beta)}{\zeta_q(0)} = 0.$$

If we take $n = 2$, since

$$\begin{aligned}\left(\varphi(x), \delta_q^\beta(x, \overline{x})\right)_{H_q^\beta} &= \int_{G_q} \tau_q^\beta(dx)\varphi(x) \int_{G_q} \mathcal{F}_q^\beta(x \cdot y)\mathcal{F}_q^\beta(\overline{x} \cdot y)\tau_q^\beta(dy) \\ &= \int_{G_q} \tau_q^\beta(dy)\mathcal{F}_q^\beta\varphi(y)\mathcal{F}_q^\beta(y\overline{x}) \\ &= \mathcal{F}_q^\beta\mathcal{F}_q^\beta\varphi(\overline{x}) \\ &= \varphi(\overline{x}),\end{aligned}$$

we have

$$\delta_q^\beta(x, \overline{x}) = \text{reproducing kernel for } H_q^\beta = \delta_{x,\overline{x}}\frac{\zeta_q(\beta)}{\zeta_q(1)}|x|_q^{-\beta}.$$

We next study the co-convolution Δ_q^β:

$$\Delta_q^\beta\varphi(x_1, x_2) = \varphi(x_1 \oplus_\beta x_2) := \int_{G_q} \varphi(x_0)\delta_q^\beta(x_0, x_1, x_2)\tau_q^\beta(dx_0).$$

Then Δ_q^β satisfies the co-commutativity relation:

$$\varphi(x_1 \oplus_\beta x_2) = \varphi(x_2 \oplus_\beta x_1)$$

and co-associativity: in fact,

$$\varphi(x_1 \oplus_\beta x_2 \oplus_\beta \cdots \oplus_\beta x_n) = \int_{G_q} \varphi(x_0)\delta_q^\beta(x_0, x_1, \ldots, x_n)\tau_q^\beta(dx_0).$$

The convolution is the dual operator of the co-convolution. It is defined by

$$\varphi_1 *_\beta \varphi_2(x_0) := \iint_{G_q \times G_q} \varphi_1(x_1)\varphi_2(x_2)\delta_q^\beta(x_0, x_1, x_2)\tau_q^\beta(dx_1)\tau_q^\beta(dx_2).$$

Then the duality of convolution and co-convolution is expressed as follows:

$$\begin{aligned}&\big(\Delta_q^\beta \varphi_0, \varphi_1 \otimes \varphi_2\big)_{H_q^\beta \otimes H_q^\beta} = (\varphi_0, \varphi_1 *_\beta \varphi_2)_{H_q^\beta}\\ &= \iiint_{G_q \times G_q \times G_q} \varphi_0(x_0)\overline{\varphi_1(x_1)\varphi_2(x_2)}\delta_q^\beta(x_0, x_1, x_2)\tau_q^\beta(dx_0)\tau_q^\beta(dx_1)\tau_q^\beta(dx_2).\end{aligned}$$

Further it holds that

$$\begin{aligned}\mathcal{F}_q^\beta(x_1 \oplus_\beta x_2) &= \int_{G_q} \tau_q^\beta(dx_0)\mathcal{F}_q^\beta(x_0)\delta_q^\beta(x_0, x_1, x_2)\\ &= \iint_{G_q \times G_q} \tau_q^\beta(dx_0)\tau_q^\beta(dy)\mathcal{F}_q^\beta(x_0)\mathcal{F}_q^\beta(x_0 \cdot y)\mathcal{F}_q^\beta(x_1 \cdot y)\mathcal{F}_q^\beta(x_2 \cdot y)\\ &= \mathcal{F}_q^\beta \mathcal{F}_q^\beta\big(\mathcal{F}_q^\beta(x_1 \cdot \_)\mathcal{F}_q^\beta(x_2 \cdot \_)\big)(\mathbf{1})\\ &= \mathcal{F}_q^\beta(x_1) \cdot \mathcal{F}_q^\beta(x_2).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\mathcal{F}_q^\beta(\varphi_1 *_\beta \varphi_2)(y) = \int_{G_q} \tau_q^\beta(dx_0)\varphi_1 *_\beta \varphi_2(x_0)\mathcal{F}_q^\beta(x_0 y)\\ &= \iiint_{G_q \times G_q \times G_q} \tau_q^\beta(dx_0)\tau_q^\beta(dx_1)\tau_q^\beta(dx_2)\varphi_1(x_1)\varphi_2(x_2)\delta_q^\beta(x_0, x_1, x_2)\mathcal{F}_q^\beta(x_0 y).\end{aligned}$$

Changing the variable $x_0 \mapsto x_0/y$, and using homogeneity

$$|y|^{-\beta}\delta_q^\beta\big(\frac{x_0}{y}, x_1, x_2\big) = \delta_q^\beta(x_0, yx_1, yx_2),$$

we have

$$\begin{aligned}\mathcal{F}_q^\beta(\varphi_1 *_\beta \varphi_2)(y) &= \iint_{G_q \times G_q} \tau_q^\beta(dx_1)\tau_q^\beta(dx_2)\varphi_1(x_1)\varphi_2(x_2)\mathcal{F}_q^\beta(yx_1 \oplus_\beta yx_2)\\ &= \iint_{G_q \times G_q} \tau_q^\beta(dx_1)\tau_q^\beta(dx_2)\varphi_1(x_1)\varphi_2(x_2)\mathcal{F}_q^\beta(yx_1)\mathcal{F}_q^\beta(yx_2)\\ &= \mathcal{F}_q^\beta\varphi_1(y) \cdot \mathcal{F}_q^\beta\varphi_2(y).\end{aligned}$$

These are the basic properties of the q-Fourier–Bessel transform, which interpolates the p-adic and real one. Note that the extra parameter β is indispensable for this interpolation.

5.3 The Basic Basis

Let us consider the space $\ell^2\big(g^{\mathbb{N}}, \tau_q^\beta\big) \subseteq H_q^\beta$ with $\tau_q^\beta(g^n) = q^{n\beta}\frac{\zeta_q(1)}{\zeta_q(\beta)}$. Note that

$$\tau_{\mathbb{Z}_q}^\beta = \phi_{\mathbb{Z}_q} \cdot \tau_q^\beta = \frac{q^{n\beta}}{\zeta_q(1+n)}\frac{\zeta_q(1)}{\zeta_q(\beta)}$$

and $\tau_{\mathbb{Z}_p}^\beta = \phi_{\mathbb{Z}_p} \cdot \tau_p^\beta$ for the γ probability measures in the p-adic and real cases. Let $H_{\mathbb{Z}_q}^\beta := \ell^2\big(g^{\mathbb{N}}, \tau_{\mathbb{Z}_q}^\beta\big)$. Then we have $\phi_{\mathbb{Z}_q} \cdot H_{\mathbb{Z}_q}^\beta = \ell^2\big(g^{\mathbb{N}}, \zeta_q(1+n)\tau_{\mathbb{Z}_q}^\beta\big)$ and

$$\phi_{\mathbb{Z}_q} \cdot H_{\mathbb{Z}_q}^\beta \subseteq \ell^2\big(g^{\mathbb{N}}, \tau_q^\beta\big) \subseteq H_{\mathbb{Z}_q}^\beta.$$

Consider the p-adic and real limit. It is easy to see that

$$\begin{array}{lccccc}
p: & H_{\mathbb{Z}_p}^\beta & \equiv & H_{\mathbb{Z}_p}^\beta & \equiv & H_{\mathbb{Z}_p}^\beta \\
 & \textcircled{p}\big\uparrow & & \textcircled{p}\big\uparrow & & \textcircled{p}\big\uparrow \\
q: & \phi_{\mathbb{Z}_q} \cdot H_{\mathbb{Z}_q}^\beta & \subseteq & \ell_2\big(g^{\mathbb{N}}, \tau_q^\beta\big) & \subseteq & H_{\mathbb{Z}_q}^\beta \\
 & \textcircled{\eta}\big\downarrow & & \textcircled{\eta}\big\downarrow & & \textcircled{\eta}\big\downarrow \\
\eta: & \phi_{\mathbb{Z}_\eta} \cdot H_{\mathbb{Z}_\eta}^\beta & \subseteq & H_\eta^\beta & \subseteq & H_{\mathbb{Z}_\eta}^\beta
\end{array}$$

Here $H_{\mathbb{Z}_p}^\beta = \ell^2\big(p^{\mathbb{N}}, \tau_{\mathbb{Z}_p}^\beta\big)$ and

$$H_\eta^\beta := L^2\Big(\mathbb{R}/\{\pm 1\}, |x|_\eta^\beta \frac{d^*x}{\zeta_\eta(\beta)}\Big), \qquad H_{\mathbb{Z}_\eta}^\beta = L^2\Big(\mathbb{R}/\{\pm 1\}, e^{-\pi x^2}|x|_\eta^\beta \frac{d^*x}{\zeta_\eta(\beta)}\Big).$$

We have also $\phi_{\mathbb{Z}_\eta} \cdot H_{\mathbb{Z}_\eta}^\beta = L^2\big(\mathbb{R}/\{\pm 1\}, e^{\pi x^2}|x|_\eta^\beta \frac{d^*x}{\zeta_\eta(\beta)}\big)$.

Now define the Basic Basis:

$$\phi_{q,m}(g^n) := \Big(\frac{\zeta_q(1)q^{m^2}}{\zeta_q(1+m)}\Big)\frac{q^{nm}}{\zeta_q(1+m+n)} \qquad (m \geq 0). \tag{5.5}$$

Note that

$$\phi_{q,0}(g^n) = \frac{1}{\zeta_q(1+n)} = \phi_{\mathbb{Z}_q}(g^n)$$

and $\phi_{q,m}$ is supported at $n \geq -m$. Taking the p-adic and real limit, we have for $m > 0$

$$\phi_{q,m} \longrightarrow \begin{cases} \phi_{p^{-m}\mathbb{Z}_p^*} & \text{for } \textcircled{p}, \\ \frac{1}{m!}(\pi x^2)^m e^{-\pi x^2} & \text{for } \textcircled{\eta}. \end{cases}$$

In fact, we have $\lim_{q\to 0}\phi_{q,m}(g^n)=\delta_{q^{-m},q^n}$, that is in the p-adic limit Ⓟ, we obtain the characteristic function on $p^{-m}\mathbb{Z}_p^*$. On the other hand taking the real limit ⓡ, we have

$$\phi_{q,m}(g^n)=\frac{\zeta_q(1)(1-q)^m}{\zeta_q(1+m)}q^{m^2}\left(\frac{q^n}{1-q}\right)^m\frac{1}{\zeta_q(1+m+n)}\to\frac{1}{m!}(\pi x^2)^m e^{-\pi x^2}.$$

Now we denote by $H^\beta_{\mathbb{Q}_q/\mathbb{Z}_q}$ the Hilbert space with the basic basis $\{\phi_{q,m}\}_{m\in\mathbb{Z}}$ as orthogonal basis with norm

$$||\phi_{q,m}||^2_{H^\beta_{\mathbb{Q}_q/\mathbb{Z}_q}}:=C^\beta_q(m)=\frac{\zeta_q(1)}{\zeta_q(1+m)}\frac{\zeta_q(\beta+m)}{\zeta_q(\beta)}q^{-m\beta}.$$

Note that $H^\beta_{\mathbb{Q}_q/\mathbb{Z}_q}$ is both a Hilbert space, and a space of functions on $g^{\mathbb{Z}}$;

$$H^\beta_{\mathbb{Q}_q/\mathbb{Z}_q}=\left\{\sum_{m\ge 0}a_m\phi_{q,m}(g^n)\,\middle|\,\sum_{m\ge 0}|a_m|^2C^\beta_q(m)<\infty\right\}\subseteq H^\beta_q.$$

We have also isometries

$$\begin{array}{ccccccc}
H^\beta_q & \supseteq & H^\beta_{\mathbb{Q}_q/\mathbb{Z}_q} & \xrightarrow{\sim} & H^\beta_{\mathbb{Z}_q} & \xrightarrow{\sim} & \phi_{\mathbb{Z}_q}\cdot H^\beta_{\mathbb{Z}_q} \subseteq H^\beta_q\\
 & & \Cup & & \Cup & & \Cup\\
 & & \phi_{q,m} & \longmapsto & \varphi^\beta_{\mathbb{Z}_q,m} & \longmapsto & \phi_{\mathbb{Z}_q}\varphi^\beta_{\mathbb{Z}_q,m}
\end{array}$$

since these orthogonal basis all have the same norm.

Let us calculate

$$\begin{aligned}
\tau^s_q(\phi_{q,m})&=\frac{\zeta_q(1)}{\zeta_q(s)}\sum_n\frac{\zeta_q(1)}{\zeta_q(1+m)}\frac{q^{m(m+n)}}{\zeta_q(1+m+n)}q^{ns}\\
&=\frac{\zeta_q(1)}{\zeta_q(1+m)}\frac{\zeta_q(s+m)}{\zeta_q(s)}q^{-ms}\\
&=C^s_q(m)
\end{aligned}$$

Recall from the theory of the Laguerre basis, we have the formula

$$\tau^s_q\big(\phi_{\mathbb{Z}_q}\cdot\varphi^\beta_{\mathbb{Z}_q,m}\big)=\tau^s_{\mathbb{Z}_q}\left(\varphi^\beta_{\mathbb{Z}_q,m}\right)=C^{\beta-s}_q(m).$$

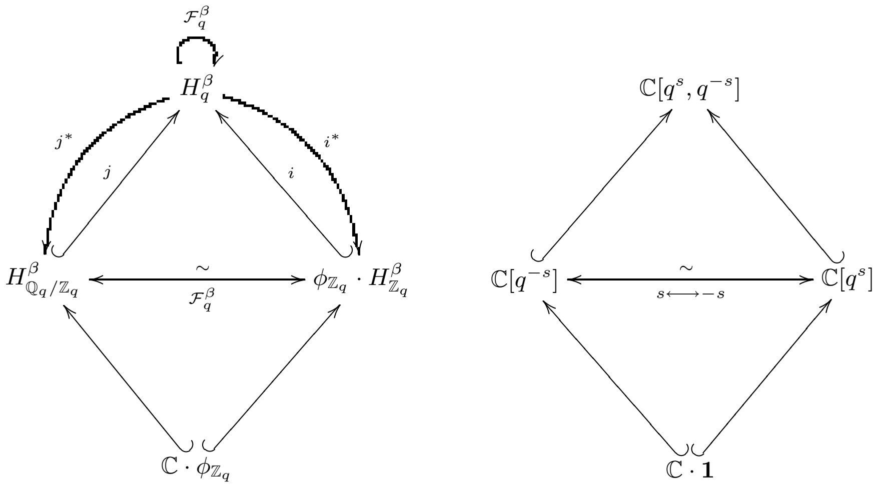
These show that

$$\tau^{\frac{\beta}{2}+s}(\phi_{q,m})=C_q^{\frac{\beta}{2}+s}(m)=\tau^{\frac{\beta}{2}-s}\big(\phi_{\mathbb{Z}_q}\cdot\varphi^\beta_{\mathbb{Z}_q,m}\big).$$

Therefore we can conclude that

$$\mathcal{F}^\beta_q(\phi_{q,m})=\phi_{\mathbb{Z}_q}\varphi^\beta_{\mathbb{Z}_q,m}$$

and the isomorphism $H^\beta_{\mathbb{Q}_q/\mathbb{Z}_q} \overset{\sim}{\longleftrightarrow} \phi_{\mathbb{Z}_q} \cdot H^\beta_{\mathbb{Z}_q}$ is given by the Fourier–Bessel transform $\mathcal{F}^\beta_q$. The following commutative diagram gives a summery for the theory of the q-Fourier Bessel transforms:



Here the map i^* (resp. j^*) is the adjoint operator of the inclusion i (resp. j) and it is given by

$$i^*\varphi = \phi_{\mathbb{Z}_q} \cdot \varphi \qquad (\text{resp. } j^*\varphi = \phi_{\mathbb{Z}_q} *_\beta \varphi).$$

We get another expansion of the kernel $\mathcal{F}^\beta_q$

$$\mathcal{F}^\beta_q(x \cdot y) = \sum_{m \geq 0} \frac{1}{C^\beta_q(m)} \phi_{q,m}(x) \varphi^\beta_{\mathbb{Z}_q,m}(y) \phi_{\mathbb{Z}_q}(y).$$

and similarly

$$\phi_{\mathbb{Z}_q}(x \oplus_\beta y) = \sum_{m \geq 0} \frac{1}{C^\beta_q(m)} \phi^\beta_{q,m}(x) \phi^\beta_{q,m}(y).$$

We have an algebra acting on H^β_q generated by X and Y;

$$X\varphi(g^n) := q^n \varphi(g^n), \qquad Y\varphi(g^n) := \varphi(g^{n-1})$$

with q-commutativity:

$$XY = qYX.$$

These are invertible operators;

$$X^{-1}\varphi(g^n) := q^{-n}\varphi(g^n), \qquad Y^{-1}\varphi(g^n) := \varphi(g^{n+1})$$

Using the q-binomial theorem, we see that $N := X + Y$ satisfies

$$N\phi_{q,m} = q^{-m}\phi_{q,m}.$$

This shows that the basic basis are eigenfunction of the operator N. Notice that we have

$$\begin{aligned} A_-\phi_{q,m} &= \phi_{q,m-1} && \text{with } A_- := Y\big[1 - x^{-1}(1-Y)\big], \\ A_+\phi_{q,m} &= [m+1]_q q^{-m}\phi_{q,m+1} && \text{with } A_+ := (1-q)^{-1}Y^{-1}X. \end{aligned}$$

The conjugation of X, Y with $\mathcal{F}_q^\beta$ are also useful:

$$\begin{aligned} \mathcal{F}_q^\beta Y \mathcal{F}_q^\beta &= q^\beta Y^{-1}, \\ \mathcal{F}_q^\beta X \mathcal{F}_q^\beta &= (1 - q^\beta Y^{-1})X^{-1}(1-Y). \end{aligned}$$

For example, in the real limit ⓷, we have

$$\begin{gathered} (1-q)^{-1}X \longmapsto \pi x^2, \\ (1-q)^{-1}(1-Y) \longmapsto -\frac{x}{2}\frac{\partial}{\partial x}, \quad (1-q)^{-1}(1-q^\beta Y^{-1}) \longmapsto \frac{\beta}{2} + \frac{x}{2}\frac{\partial}{\partial x}, \end{gathered}$$

and conjugation with $\mathcal{F}_\eta^\beta$ gives

$$\begin{aligned} \mathcal{F}_\eta^\beta\Big(-\frac{x}{2}\frac{\partial}{\partial x}\Big)\mathcal{F}_\eta^\beta &= \frac{\beta}{2} + \frac{x}{2}\frac{\partial}{\partial x}, \\ \mathcal{F}_\eta^\beta(\pi x^2)\mathcal{F}_\eta^\beta &= \Big(\frac{\beta}{2} + \frac{x}{2}\frac{\partial}{\partial x}\Big)\frac{1}{\pi x^2}\Big(-\frac{x}{2}\frac{\partial}{\partial x}\Big) \\ &= -\frac{1}{4\pi}\Big(\frac{\partial^2}{\partial x^2} + \frac{\beta-1}{x}\frac{\partial}{\partial x}\Big) \\ &= -\frac{1}{4\pi}x^{1-\beta}\frac{\partial}{\partial x}x^{\beta-1}\frac{\partial}{\partial x}. \end{aligned}$$

We call the algebra generated by N, A_+ and A_- the "q-Heisenberg algebra". We have the relations

$$A_\pm N = q^{\pm 1}NA_\pm, \quad A_-A_+ - A_+A_- = N \qquad (\mathfrak{sl}_2\text{-relations})$$

and an extra relation

$$A_-A_+ - q^{-1}A_+A_- = \mathrm{id}. \tag{5.6}$$

This extra relation (5.6) explains the Heisenberg relation up the ladder for the boundary space of the Markov chain obtained before;

$$\begin{array}{ccccc}
H_{\mathbb{Z}}^{\beta} & \underset{D_{\beta}^{+}}{\overset{D}{\rightleftarrows}} & H_{\mathbb{Z}}^{\beta+1} & \underset{D_{\beta+1}^{+}}{\overset{D}{\rightleftarrows}} & H_{\mathbb{Z}}^{\beta+2} \\
\phi_{\mathbb{Z}}\cdot \downarrow \wr & & \phi_{\mathbb{Z}}\cdot \downarrow \wr & & \phi_{\mathbb{Z}}\cdot \downarrow \wr \\
\phi_{\mathbb{Z}} \cdot H_{\mathbb{Z}}^{\beta} & & \phi_{\mathbb{Z}} \cdot H_{\mathbb{Z}}^{\beta+1} & & \phi_{\mathbb{Z}} \cdot H_{\mathbb{Z}}^{\beta+2} \\
\mathcal{F}^{\beta} \updownarrow \wr & & \mathcal{F}^{\beta+1} \updownarrow \wr & & \mathcal{F}^{\beta+2} \updownarrow \wr \\
H_{\mathbb{Q}/\mathbb{Z}}^{\beta} & \underset{A_{+}}{\overset{\frac{q^{-\beta}}{1-q}A_{-}}{\rightleftarrows}} & H_{\mathbb{Q}/\mathbb{Z}}^{\beta+1} & \underset{A_{+}}{\overset{\frac{q^{-(\beta+1)}}{1-q}A_{-}}{\rightleftarrows}} & H_{\mathbb{Q}/\mathbb{Z}}^{\beta+2}
\end{array}$$

Then the Heisenberg relation

$$DD_{\beta}^{+} - D_{\beta+1}^{+} D = \frac{q^{-\beta}}{1-q} \mathrm{id}_{H_{\mathbb{Z}}^{\beta+1}}, \tag{5.7}$$

when written by using the diagram above in terms of the operators A_+ and A_-, translates into the extra relation (5.6).

6

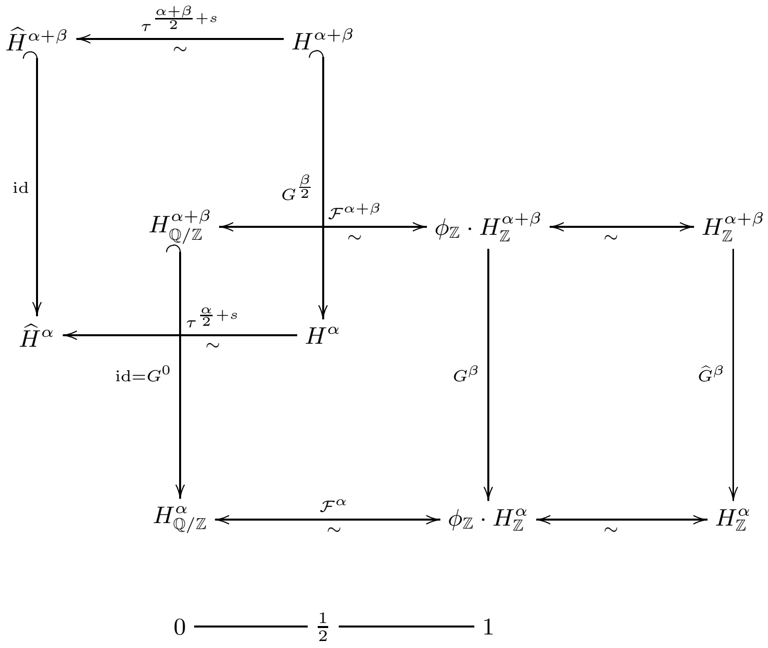
Pure Basis and Semi-Group

Summary. In Sect. 6.1 we obtain the basis for the total space H^β from the basis of its subspace $H^\beta_{\mathbb{Z}}$ and $H^\beta_{\mathbb{Q}/\mathbb{Z}}$. A remarkable phenomenom occurs in the real case; with $\varphi^\beta_m(x) = L_m^{\frac{\beta}{2}-1}(\pi x^2)$ the Laguerre basis for $H^\beta_{\mathbb{Z}_\eta}$,

$$\tau^s_{\mathbb{Z}_\eta}(\varphi^\beta_m(x)) = (-1)C^{\beta-s}_\eta(m) \text{ : a polynomial with zeros at } s = \beta, \beta+2, \ldots, \beta+2(m-1),$$

$$\tau^s_{\mathbb{Z}_\eta}(\varphi^\beta_m(\sqrt{2}x)) \text{ : a polynomial with zeros at } \mathrm{Re}(s) = \frac{\beta}{2}.$$

In Sect. 6.2 we give the semi-group G^β with $\tau^s(G^\beta\varphi) = \tau^{s+\beta}(\varphi)$ for q, p and η. The picture can be summarized in the diagram:



We notice that for q and p, $\widehat{G}^\beta$ is given by convolution with the measures of the finite layers of the γ-chain.

In Sect. 6.3 we give the global semi-group $G^\beta_{\mathbb{Q}}$, and describe its basic properties very briefly (this is the only chapter where we do global theory in this book). In particular, we point the relevance of $G^\beta_{\mathbb{Q}}$ to the explicit sums, and give an equivalence of the Riemann hypothesis in terms of $G^\beta_{\mathbb{Q}}$.

6.1 The Pure Basis

Let us specialize the Sect. 5.3 to the case of η. We have the Laguerre basis

$$\varphi^\beta_{\mathbb{Z}_\eta,m} = \varphi^\beta_m = (-1)^m L_m^{\frac{\beta}{2}-1}(\pi x^2).$$

This is the orthogonal basis for $H^\beta_{\mathbb{Z}_\eta} = L^2\big(\mathbb{R}/\{\pm 1\}, e^{-\pi x^2}|x|^\beta_\eta \frac{d^*x}{\zeta_\eta(\beta)}\big)$. Then

$$\varphi^\beta_m(x)e^{-\frac{\pi x^2}{2}}$$

is an orthogonal basis for $H^\beta_\eta = L^2\big(\mathbb{R}/\{\pm 1\}, |x|^\beta_\eta \frac{d^*x}{\zeta_\eta(\beta)}\big)$. To normalize the Gaussian, we replace x by $\sqrt{2}x$ in the basis $\varphi^\beta_m(x)e^{-\frac{\pi x^2}{2}}$. Namely, we put

$$\Phi^\beta_m(x) := \varphi^\beta_m(\sqrt{2}x)e^{-\pi x^2} = (-1)^m L_m^{\frac{\beta}{2}-1}(2\pi x^2)e^{-\pi x^2}.$$

Then $\Phi^\beta_m(x)$ is also an orthogonal basis for H^β_η.

Let us go to the analytic space via $\tau_\eta^{\frac{\beta}{2}+s}$, where the infinitesimal generator of the action of the multiplicative groups $\big(\frac{\beta}{2} + x\frac{\partial}{\partial x}\big)$ corresponds to multiplication by $-s$:

$$\begin{array}{ccccccc}
\mathrm{Span}\{\Phi^\beta_m(x)\} & = & \mathbb{C}[x]e^{-\pi x^2} & = & \mathrm{Span}\big\{\big(\frac{\beta}{2}+x\frac{\partial}{\partial x}\big)^n e^{-\pi x^2}\big\} & \subseteq & H^\beta_\eta \\
\updownarrow \wr & & & & \updownarrow \wr & & \tau_\eta^{\frac{\beta}{2}+s} \updownarrow \wr \\
\mathrm{Span}\{\widehat{f}^\beta_m(s)\} & = & \mathbb{C}[s] & = & \mathrm{Span}\{s^n \cdot \mathbf{1}\} & \subseteq & \widehat{H}^\beta_\eta
\end{array}$$

Here $\widehat{H}^\beta_\eta := L^2\big(i\mathbb{R}, \zeta_\eta\big(\frac{\beta}{2}+s, \frac{\beta}{2}-s\big)d^\circ s\big)$ is the dual space of H^β_η and

$$\widehat{f}^\beta_m(s) := \tau_\eta^{\frac{\beta}{2}+s}\big(\Phi^\beta_m\big)$$

is the orthogonal basis for $\widehat{H}^\beta_\eta$, which corresponds to $\Phi^\beta_m(x)$. This is equivalent to

$$\Phi^\beta_m = \widehat{f}^\beta_m\Big(\frac{\beta}{2} + x\frac{\partial}{\partial x}\Big)e^{-\pi x^2}.$$

Since the measure $\zeta_\eta\big(\frac{\beta}{2}+s, \frac{\beta}{2}-s\big)d^\circ s$ is symmetric in s and $-s$, we immediately see that, depending on m, $\widehat{f}^\beta_m(s)$ is odd or even;

$$\widehat{f}^{\beta}_{m}(-s) = (-1)^m \widehat{f}^{\beta}_{m}(s).$$

Hence we have

$$\mathcal{F}^{\beta}_{\eta}\Phi^{\beta}_{m} = (-1)^m \Phi^{\beta}_{m}.$$

The recursion relation

$$-s\widehat{f}^{\beta}_{m}(s) = (m+1)\widehat{f}^{\beta}_{m+1}(s) - \Big(m-1+\frac{\beta}{2}\Big)\widehat{f}^{\beta}_{m-1}(s), \tag{6.1}$$

is equivalent to

$$\Big(\frac{\beta}{2} + x\frac{\partial}{\partial x}\Big)\Phi^{\beta}_{m}(x) = (m+1)\Phi^{\beta}_{m+1}(x) - \Big(m-1+\frac{\beta}{2}\Big)\Phi^{\beta}_{m-1}(x).$$

This recursion (6.1) is also equivalent to

$$\begin{aligned}\widehat{f}^{\beta}_{m}(s) &= \frac{(-1)^m}{m!}\cdot\det\begin{pmatrix} s & \frac{\beta}{2} & 0 & \cdots & 0 \\ 1 & s & \frac{\beta}{2}+1 & \ddots & \vdots \\ 0 & 2 & s & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\beta}{2}+m-2 \\ 0 & \cdots & 0 & m-1 & s \end{pmatrix} \\ &= \frac{(-1)^m}{m!}\cdot\det\Big[\mathrm{pr}^{\beta}_{m}\Big(s - (\frac{\beta}{2} + x\frac{\partial}{\partial x})\Big)\mathrm{pr}^{\beta}_{m}\Big],\end{aligned}$$

where

$$\mathrm{pr}^{\beta}_{m} : H^{\beta}_{\eta} \longrightarrow \bigoplus_{0\le n<m} \mathbb{C}\cdot\Phi^{\beta}_{m}$$

is the orthogonal projection (this is called the Jacobi matrix). Note that the zeros of the orthogonal polynomial $\widehat{f}^{\beta}_{m}(s)$ of degree m are located in the line $i\mathbb{R}$ (in general, the zeros of the orthogonal polynomials are always contained in the line of orthogonality). Since

$$\widehat{f}^{\beta}_{m}\Big(s-\frac{\beta}{2}\Big) = \tau^{s}_{\eta}(\Phi^{\beta}_{m}) = \int_0^\infty L_m^{\frac{\beta}{2}-1}(2\pi x^2)e^{-\pi x^2}|x|^s_\eta \frac{d^*x}{\zeta_\eta(s)}, \tag{6.2}$$

the zeros of this function of s are contained in the vertical line $\{s\in\mathbb{C} \,|\, \mathrm{Re}(s) = \frac{\beta}{2}\}$. Similarly, since

$$\begin{aligned}\tau^{s}_{\mathbb{Z}_\eta}(\varphi^{\beta}_{m}) &= \int_0^\infty L_m^{\frac{\beta}{2}-1}(\pi x^2)e^{-\pi x^2}|x|^s_\eta \frac{d^*x}{\zeta_\eta(s)} \\ &= (-1)^m C^{\beta-s}_{\eta}(m) \\ &= \frac{(-1)^m}{m!}\Big(\frac{\beta-s}{2}\Big)\Big(\frac{\beta-s}{2}+1\Big)\cdots\Big(\frac{\beta-s}{2}+m-1\Big),\end{aligned} \tag{6.3}$$

the zeros of $\tau^s_{\mathbb{Z}_\eta}(\varphi^\beta_m)$ are contained in $s \in \{\beta, \beta+2, \ldots, \beta+2(m-1)\}$. Remark that the zeros of $\tau^s_\eta(\Phi^\beta_m)$ are located in the vertical line while the one of $\tau^s_{\mathbb{Z}_\eta}(\varphi^\beta_m)$ are in the horizontal line (but the only difference between (6.2) and (6.3) is the factor 2). For the cases of $\beta = 1$, the fact that the zeros of $\widehat{f}^1_m(s)$ are contained in the line $\mathrm{Re}(s) = 1/2$ is noticed by Bump and K-S Ng (see [BN]).

We have again the ladder

$$H^\beta_\eta \underset{D^+_\beta}{\overset{D^-}{\rightleftarrows}} H^{\beta+2}_\eta \underset{D^+_{\beta+2}}{\overset{D^-}{\rightleftarrows}} H^{\beta+4}_\eta$$

Here

$$D^+_\beta = \pi x^2 - \Big(\frac{\beta}{2} + x\frac{\partial}{\partial x}\Big), \qquad D^- = \frac{1}{2}\Big(1 + \frac{1}{2\pi x}\frac{\partial}{\partial x}\Big).$$

Note that $D^+_\beta = \beta(D^-)^*$ and is the same operator as $D^+_\beta : H^{\beta+2}_{\mathbb{Z}_\eta} \to H^\beta_{\mathbb{Z}_\eta}$ of Sect. 4.5. These also satisfy the Heisenberg relation up the ladder:

$$D^- D^+_\beta - D^+_{\beta+2} D^- = \mathrm{id}_{H^{\beta+2}_\eta}.$$

Then the basis $\Phi^\beta_m(x)$ for H^β_η can be written as

$$\Phi^\beta_m = \frac{(-1)^m}{m!}(D^+)^m \cdot e^{-\pi x^2}.$$

Since D^+_β is the same operator, we have also an expression of the Laguerre basis φ^β_m:

$$\varphi^\beta_m = \frac{(-1)^m}{m!}(D^+)^m \cdot \mathbf{1}_{H^\beta_{\mathbb{Z}_\eta}}.$$

We can translate the ladder to the analytic spaces

$$\begin{array}{ccccc}
H^\beta_\eta & \underset{D^+_\beta}{\overset{D^-}{\rightleftarrows}} & H^{\beta+2}_\eta & \underset{D^+_{\beta+2}}{\overset{D^-}{\rightleftarrows}} & H^{\beta+4}_\eta \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\widehat{H}^\beta_\eta & \underset{\widehat{D}^+_\beta}{\overset{\widehat{D}^-}{\rightleftarrows}} & \widehat{H}^{\beta+2}_\eta & \underset{\widehat{D}^+_{\beta+2}}{\overset{\widehat{D}^-}{\rightleftarrows}} & \widehat{H}^{\beta+4}_\eta
\end{array}$$

Here

$$\widehat{D}^+_\beta \widehat{f}(s) := \frac{1}{2}\Big[\Big(\frac{\beta}{2} + s\Big)\widehat{f}(s+1) - \Big(\frac{\beta}{2} - s\Big)\widehat{f}(s-1)\Big],$$
$$\widehat{D}^- \widehat{f}(s) := \frac{1}{2}\big[\widehat{f}(s+1) - \widehat{f}(s-1)\big].$$

Also we have the Heisenberg relation

$$\widehat{D}^-\widehat{D}^+_\beta - \widehat{D}^+_{\beta+2}\widehat{D}^- = \mathrm{id}_{\widehat{H}^{\beta+2}_\eta}.$$

Then, using this relation, one can prove by induction that $\widehat{f}^\beta_m(s)$ have all its zeros on the line $\mathrm{Re}(s) = 0$ (this is why we call Φ^β_m the "pure basis"). Similarly, we have the expression

$$\widehat{f}^\beta_m = \frac{(-1)^m}{m!}(\widehat{D}^+)^m \cdot \mathbf{1}_{\widehat{H}^\beta_\eta}.$$

By induction, we have explicitly

$$\begin{aligned}\widehat{f}^\beta_m &= \frac{(\frac{\beta}{2})_m}{m!}\,{}_2F_1\Big(-m, \frac{\beta}{4} + \frac{s}{2}; \frac{\beta}{2}; 2\Big) \\ &= \frac{\pi^m}{m!}\sum_{m\geq 0}\binom{m}{k}(-1)^k\frac{\zeta_\eta(\beta+2m)}{\zeta_\eta(\beta+2k)}\frac{\zeta_\eta\big(\frac{\beta}{2}+2k+s\big)}{\zeta_\eta\big(\frac{\beta}{2}+s\big)}2^k.\end{aligned}$$

These are called the Meixner–Pollaczeck polynomials.

We next consider the p-adic pure basis. Let

$$\begin{aligned}\Phi^\beta_0 &:= \phi_{\mathbb{Z}_p}, \\ \Phi^\beta_{2m-\delta} &:= \phi_{p^{-m}\mathbb{Z}_p^*} + (-1)^\delta\varphi^\beta_{\mathbb{Z}_p,m} = \big(1 + (-1)^\delta\mathcal{F}^\beta\big)\phi_{m,p} \quad (m > 0),\end{aligned}$$

where $\delta \in \{0, 1\}$. This is an orthogonal basis for $H^\beta_p = L^2\big(\mathbb{Q}_p/\mathbb{Z}_p^*, |x|^\beta_p\frac{d^*x}{\zeta_p(\beta)}\big)$. In the analytic space, we put

$$\begin{aligned}\widehat{f}^\beta_{2m-\delta}(s) :&= \tau_p^{\frac{\beta}{2}+s}(\Phi^\beta_{2m-\delta}) \\ &= C_p^{\frac{\beta}{2}+s}(m) + (-1)^\delta C_p^{\frac{\beta}{2}-s}(m).\end{aligned}$$

Recall that $C^s_p(m) = (1 - p^{-s})p^{ms}$. Then it is easy to see that

$$\widehat{f}^\beta_m(-s) = (-1)^m\widehat{f}^\beta_m(s), \qquad \mathcal{F}^\beta_p\Phi^\beta_m = (-1)^m\Phi^\beta_m.$$

The q-theory for Φ^β_m and $\widehat{f}^\beta_m(s)$ which interpolates the p-adic and real one is a special case of the Askey–Wilson polynomials.

6.2 The Semi-Group G^β

We next study the function G^β_q on $G_q = g^\mathbb{Z}$ defined by

$$G^\beta_q(g^n) := \frac{\zeta_q(\beta+n)}{\zeta_q(1+n)}.$$

Notice that G_q^β is supported on g^n for $n \geq 0$. Let us calculate the p-adic and the real limit. First of all, for the limit ⓟ, we see that

$$G_p^\beta(p^n\mathbb{Z}_p^*) := \lim_{\substack{q\to 0\\ q^\beta \to p^{-\beta}}} G_q^\beta(g^n) = \begin{cases} 0 & \text{for } n<0,\\ (1-p^{-\beta})^{-1} & \text{for } n=0,\\ 1 & \text{for } n>0. \end{cases}$$

On the other hand, for the complex η limit, we have

$$\begin{aligned} G_\eta^\beta(x) : &= \lim_{\substack{q\to 1,\ n\to\infty\\ q^n \to |x|^2}} G_q^\beta(g^n) \\ &= \lim_{\substack{q\to 1,\ n\to\infty\\ q^n \to |x|^2}} \sum_{k\geq 0} \frac{\zeta_q(1-\beta+k)}{\zeta_q(1-\beta)} \frac{\zeta_q(1)}{\zeta_q(1+k)} q^{(\beta+n)k} \\ &= \lim_{\substack{q\to 1,\ n\to\infty\\ q^n \to |x|^2}} \sum_{k\geq 0} \frac{(1-q^{1-\beta})(1-q^{2-\beta})\cdots(1-q^{k-\beta})}{(1-q)(1-q^2)\cdots(1-q^k)} q^{\beta k} q^{\beta n} \\ &= \sum_{k\geq 0} \frac{(1-\beta)(2-\beta)\cdots(k-\beta)}{k!} |x|_\eta^{2k} \\ &= \left(1-|x|_\eta^2\right)^{\beta-1}. \end{aligned}$$

For the real η case, we always replace β by $\beta/2$. Hence we have

$$G_\eta^\beta(x) = \left(1-|x|_\eta^2\right)_+^{\frac{\beta}{2}-1} := \begin{cases} 0 & \text{for } |x|_\eta > 1,\\ \left(1-|x|_\eta^2\right)^{\frac{\beta}{2}-1} & \text{for } |x|_\eta \leq 1. \end{cases}$$

The basic property of G^β for all p, η and q is given by

$$\int_G G^\beta(x)|x|^s d^*x = \zeta(s,\beta). \tag{6.4}$$

In fact, for the case of q, we have from the q-β-sum,

$$\int_{G_q} G_q^\beta(x)|x|_q^s d^*x = \sum_{n\geq 0} \frac{\zeta_q(\beta+n)}{\zeta_q(1+n)} q^{ns} = \zeta_q(s,\beta).$$

Hence we obtain the formula (6.4). Then, taking the p-adic and η limit, we have (6.4) for the case of p and η.

Now we define the operator G^β, which is the same symbol G^β as above, by

$$G^\beta\varphi(y) := \int_G \tau^\beta(dx) G^\beta(y/x)\varphi(x).$$

For example, for the case of q, we have

$$\begin{aligned} G_q^\beta\varphi(g^n) &= \sum_m q^{m\beta} \frac{\zeta_q(1)}{\zeta_q(\beta)} \frac{\zeta_q(\beta+n-m)}{\zeta_q(1+n-m)} \varphi(g^m) \\ &= q^{\beta n} \sum_m C_q^\beta(n-m)\varphi(g^m). \end{aligned}$$

Then let us calculate the Tate distribution of $G^\beta\varphi$;

$$\tau^s(G^\beta\varphi) = \tau^{s+\beta}(\varphi). \tag{6.5}$$

Actually, since

$$\int_G \frac{d^*y|y|^s}{\zeta(s)} G^\beta(y) = \frac{\zeta(\beta)}{\zeta(\beta+s)},$$

we have

$$\begin{aligned}\tau^s(G^\beta\varphi) &= \int \frac{d^*y|y|^s}{\zeta(s)} \int \frac{d^*x|x|^\beta}{\zeta(\beta)} G^\beta(y/x)\varphi(x) \\ &= \int \frac{d^*x}{\zeta(\beta)}|x|^{\beta+s}\Big[\int \frac{d^*y|y|^s}{\zeta(s)} G^\beta(y)\Big]\varphi(y) \\ &= \tau^{s+\beta}(\varphi).\end{aligned}$$

Hence we obtain the formula (6.5). Therefore $\{G^\beta\}$ is a semi-group, that is,

$$\begin{aligned}&G^{\beta_1}G^{\beta_2} = G^{\beta_1+\beta_2}, \\ &\lim_{\beta\to 0} G^\beta = \text{id}.\end{aligned}$$

Note also that the cyclic vector $\phi_{\mathbb{Z}}$ is characterized by

$$\tau^s(\phi_{\mathbb{Z}}) = 1, \quad \text{and hence} \quad G^\beta\phi_{\mathbb{Z}} = \phi_{\mathbb{Z}}.$$

For example, for the case of η, we have

$$\begin{aligned}G^\beta_\eta\varphi(y) &= \int_{\mathbb{R}+} \frac{d^*x}{\zeta_\eta(\beta)}|x|^\beta_\eta\big(1 - \big|y/x\big|^2_\eta\big)_+^{\frac{\beta}{2}-1}\varphi(x) \\ &= \frac{1}{\zeta_\eta(\beta)}\int_{\mathbb{R}+} dx\cdot x\big(|x|^2_\eta - |y|^2_\eta\big)_+^{\frac{\beta}{2}-1}\varphi(x).\end{aligned}$$

This shows that the operator G^β_η is just the fractional differentiation modulo the change of the variables $x \mapsto \sqrt{x}$ and $y \mapsto \sqrt{y}$.

Next we regard G^β as the map

$$G^{\frac{\beta}{2}} : H^{\alpha+\beta} \longrightarrow H^\alpha$$

It is easy to check that $G^{\frac{\beta}{2}}$ intertwines the representation π:

$$G^{\frac{\beta}{2}}\pi^{\alpha+\beta}(a) = \pi^\alpha(a)G^{\frac{\beta}{2}},$$

and it preserves the cyclic vectors. This also follows from (6.5) which gives the following commutative diagram:

$$\begin{array}{ccc} H^{\alpha+\beta} & \xrightarrow{\;G^{\frac{\beta}{2}}\;} & H^{\alpha} \\ {\scriptstyle \tau^{\frac{\alpha+\beta}{2}+s}}\Big\downarrow\wr & & \wr\Big\downarrow{\scriptstyle \tau^{\frac{\alpha}{2}+s}} \\ \widehat{H}^{\alpha+\beta} & \xrightarrow{\;\mathrm{id}\;} & \widehat{H}^{\alpha} \end{array}$$

Notice also that (6.5) gives

$$\mathcal{F}^{\alpha}(G^{\beta}\varphi) = \mathcal{F}^{\alpha+\beta}(\varphi)$$

since

$$\tau^{s}(\mathcal{F}^{\alpha}G^{\beta}\varphi) = \tau^{\alpha-s}(G^{\beta}\varphi) = \tau^{\alpha+\beta-s}(\varphi) = \tau^{s}(\mathcal{F}^{\alpha+\beta}\varphi).$$

Then, for all p, η and q, we have the following commutative diagram:

$$\begin{array}{ccc} \phi_{\mathbb{Z}}\cdot H_{\mathbb{Z}}^{\alpha+\beta} & \xrightarrow{\;G^{\beta}\;} & \phi_{\mathbb{Z}}\cdot H_{\mathbb{Z}}^{\alpha} \\ {\scriptstyle \mathcal{F}^{\alpha+\beta}}\Big\updownarrow\wr & & \wr\Big\updownarrow{\scriptstyle \mathcal{F}^{\alpha}} \\ H_{\mathbb{Q}/\mathbb{Z}}^{\alpha+\beta} & \xrightarrow[\;\mathrm{id}\;]{} & H_{\mathbb{Q}/\mathbb{Z}}^{\alpha} \end{array}$$

Since the basic basis is map via $\mathcal{F}^{\alpha}$ to the Laguerre basis $\varphi_{\mathbb{Z},m}^{\alpha}$ multiplied by $\phi_{\mathbb{Z}}$, we conclude that

$$G^{\beta}(\phi_{\mathbb{Z}}\cdot\varphi_{\mathbb{Z},n}^{\alpha+\beta}) = \phi_{\mathbb{Z}}\cdot\varphi_{\mathbb{Z},n}^{\alpha}.$$

We also can view the operator G^{β} as an operator on the boundary space $H_{\mathbb{Z}}^{\alpha+\beta}$ of the Markov chain.

Let $\widehat{G}^{\beta} := \phi_{\mathbb{Z}}^{-1}\cdot G^{\beta}\cdot\phi_{\mathbb{Z}} : H_{\mathbb{Z}}^{\alpha+\beta} \to H_{\mathbb{Z}}^{\alpha}$. Then we have

$$\widehat{G}^{\beta}\varphi_{\mathbb{Z},n}^{\alpha+\beta} = \varphi_{\mathbb{Z},n}^{\alpha}.$$

For example, for the case of q, we have

$$\begin{aligned} \widehat{G}_q^{\beta}\varphi(g^n) &= \zeta_q(1+m)\sum_{n\geq 0} q^{n\beta}\frac{\zeta_q(1)}{\zeta_q(\beta)}\frac{\zeta_q(\beta+m-n)}{\zeta_q(1+m-n)}\frac{1}{\zeta_q(1+n)}\varphi(g^n) \\ &= \sum_{0\leq n\leq m}\Big[\begin{bmatrix} m \\ n \end{bmatrix}_q q^{n\beta}\frac{\zeta_q(\beta+m-n)}{\zeta_q(1+m-n)}\Big]\varphi(g^n) \end{aligned}$$

$$= \sum_{0\le n\le m} \tau^{\beta}_{q(m)}(n, m-n)\varphi(g^n).$$

Here we use the relation

$$\begin{bmatrix} m \\ n \end{bmatrix}_q q^{n\beta} \frac{\zeta_q(\beta+m-n)}{\zeta_q(1+m-n)} = \tau^{\beta}_{q(m)}(n, m-n)$$

and $\tau^{\beta}_{q(m)}$ is just the measure of the m-th layer of the q-β-Markov chain.

6.3 Global Tate–Iwasawa Theory

Next we study the global theory. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$. The global Tate measure $\tau^{\beta}_{\mathbb{A}}$ is defined by

$$\tau^{\beta}_{\mathbb{A}} := \bigotimes_{p\ge\eta} \tau^{\beta}_p \in \mathcal{S}^*(\mathbb{A})^{\widehat{\mathbb{Z}}^*_{\mathbb{A}}\mathbb{Q}^*},$$

where $\mathcal{S}^*(\mathbb{A})$ is the space of distributions on the adeles, the dual space of $\mathcal{S}(\mathbb{A}) := \bigotimes_{p\ge\eta} \mathcal{S}(\mathbb{Q}_p)$ with respect to $\phi_{\mathbb{Z}_p}$; and we are interested only in $\mathbb{Q}^*$-invariant distributions; also, for simplicity, we concentrate on $\zeta(s)$ rather than on L-functions and take $\widehat{\mathbb{Z}}^*_{\mathbb{A}} := \prod_{p\ge\eta} \mathbb{Z}^*_p$-invariant distributions. For $\beta > 0$, $\tau^{\beta}_{\mathbb{A}}$ gives a positive measure on $\mathbb{A}$. Hence we have the Hilbert space

$$H^{\beta}_{\mathbb{A}} := L^2(\mathbb{A}, \tau^{\beta}_{\mathbb{A}})^{\widehat{\mathbb{Z}}^*\{\pm 1\}} = \bigotimes_{p\ge\eta} H^{\beta}_p \quad \text{with respect to } \phi_{\mathbb{Z}_p}.$$

We take the cyclic vector $\phi_{\mathbb{Z}_{\mathbb{A}}} := \prod_{p\ge\eta} \phi_{\mathbb{Z}_p}$. Let $\mathbb{A}^*$ be the idele group and $|\cdot|_{\mathbb{A}} := \prod_{p\ge\eta} |\cdot|_p$ be the global absolute value. Consider the unitary representation

$$\pi^{\beta}_{\mathbb{A}} : \mathbb{A}^*/\widehat{\mathbb{Z}}^*\{\pm 1\} \longrightarrow U(H^{\beta}_{\mathbb{A}}); \quad \pi^{\beta}_{\mathbb{A}}(a)\varphi(x) = |a|_{\mathbb{A}}^{-\frac{\beta}{2}} \varphi(a^{-1}x),$$

Then the dual Hilbert space $\widehat{H}^{\beta}_{\mathbb{A}}$ of $H^{\beta}_{\mathbb{A}}$ is given by

$$\begin{aligned} \widehat{H}^{\beta}_{\mathbb{A}} :&= L^2\Big(\big(\mathbb{A}^*/\widehat{\mathbb{Z}}^*\{\pm 1\}\big)^{\wedge}, \bigotimes_{p\ge\eta} \zeta_p\Big(\frac{\beta}{2} + it_p, \frac{\beta}{2} - it_p\Big) d^{\circ}t_p \Big) \\ &= \bigotimes_{p\ge\eta} \widehat{H}^{\beta}_p \quad \text{with respect to } \mathbf{1}. \end{aligned}$$

We again obtain the decomposition of the representation $\pi^{\beta}_{\mathbb{A}}$

$$H^{\beta}_{\mathbb{A}} \stackrel{\sim}{\longrightarrow} \widehat{H}^{\beta}_{\mathbb{A}}.$$

We have the global Fourier–Bessel transforms $\mathcal{F}_{\mathbb{A}}^{\beta} := \bigotimes_{p \geq \eta} \mathcal{F}_p^{\beta}$ on the space $H_{\mathbb{A}}^{\beta}$ which intertwines $\pi_{\mathbb{A}}^{\beta}(a)$ and $\pi_{\mathbb{A}}^{\beta}(a^{-1})$ and it corresponds to $\widehat{\mathcal{F}}_{\mathbb{A}}^{\beta}$ on $\widehat{H}_{\mathbb{A}}^{\beta}$ which transform t to $-t$ for $t \in (\mathbb{A}^*/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\{\pm 1\})^\wedge$. Similarly, we obtain the spaces

$$H_{\mathbb{Z}_{\mathbb{A}}}^{\beta} := \bigotimes_{p \geq \eta} H_{\mathbb{Z}_p}^{\beta}, \qquad H_{\mathbb{Q}_{\mathbb{A}}/\mathbb{Z}_{\mathbb{A}}}^{\beta} := \bigotimes_{p \geq \eta} H_{\mathbb{Q}_p/\mathbb{Z}_p}^{\beta}.$$

The important thing is that, in the measure $\bigotimes_{p \geq \eta} \zeta_p\Big(\frac{\beta}{2} + it_p, \frac{\beta}{2} - it_p\Big) d^\circ t_p$ on $\widehat{H}_{\mathbb{A}}^{\beta}$,

$$t = \{t_p\}_{p \geq \eta} \in (\mathbb{A}^*/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\{\pm 1\})^\wedge$$

and

$$(\mathbb{A}^*/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\{\pm 1\})^\wedge = \prod_{p \geq \eta} (\mathbb{Q}_p^*/\mathbb{Z}_p^*)^\wedge = \mathbb{R} \times \prod_{p > \eta} \mathbb{R}/\frac{2\pi}{\log p}\mathbb{Z} \qquad (\text{“infinite torus”}).$$

Notice that, corresponding to the homomorphism given by the absolute value

$$\mathbb{A}^*/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\{\pm 1\} \xrightarrow{|\cdot|_{\mathbb{A}}} \mathbb{A}^*/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\mathbb{Q}^* = \mathbb{R}^+,$$

we have the dual homomorphism given by the diagonal embedding

$$(\mathbb{R}^+)^\wedge = i\mathbb{R} \longrightarrow i\mathbb{R} \times \prod_{p > \eta} i\mathbb{R}/\frac{2\pi i}{\log p}\mathbb{Z} = \big(\mathbb{A}^*/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\{\pm 1\}\big)^\wedge.$$

More generally, taking the dual group of all quasi-characters, that is, taking $\mathrm{Hom}(\cdot, \mathbb{C}^*)$ instead of $\mathrm{Hom}(\cdot, \mathbb{C}^{(1)})$, we have the diagonal embedding

$$\mathbb{C} \longrightarrow \mathbb{C} \times \prod_{p > \eta} \mathbb{C}/\frac{2\pi i}{\log p}\mathbb{Z}$$

The very definition of the global zeta function as Euler-product is just the restriction along this diagonal embedding:

$$\zeta_{\mathbb{A}}(s) := \prod_{p \geq \eta} \zeta_p(s) = \Big(\prod_{p \geq \eta} \zeta_p(s_p)\Big)\Big|_{s_p = s},$$

$$\zeta_{\mathbb{A}}(s,t) := \prod_{p \geq \eta} \zeta_p(s,t) = \Big(\prod_{p \geq \eta} \zeta_p(s_p, t_p)\Big)\Big|_{s_p = s, t_p = t} = \frac{\zeta_{\mathbb{A}}(s)\zeta_{\mathbb{A}}(t)}{\zeta_{\mathbb{A}}(s+t)}.$$

Thus the meaning in the global part of the Tate–Iwasawa theory of the restriction to the characters which are trivial on $\mathbb{Q}^*$ is that we are working with the action of $\mathbb{A}^*/\mathbb{Q}^*$ on the space $\mathbb{A}/\mathbb{Q}^*$, or for zeta, with the action of $\mathbb{R}^+ = \mathbb{A}^*/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\mathbb{Q}^*$ on the space $\mathbb{A}/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\mathbb{Q}^*$:

$$\pi_{\mathbb{A}/\mathbb{Q}^*}^{\beta} : \mathbb{R}^+ = \mathbb{A}^*/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\mathbb{Q}^* :\longrightarrow U\big(L^2(\mathbb{A}/\widehat{\mathbb{Z}}_{\mathbb{A}}^*\mathbb{Q}^*, \tau_{\mathbb{A}}^{\beta})\big).$$

For $\beta > 1$, the Tate measure $\tau_{\mathbb{A}}^{\beta}$ is supported on $\mathbb{A}^*$ and the space

$$H^{\beta}_{\mathbb{A}/\mathbb{Q}^*} := L^2\big(\mathbb{A}^*/\widehat{\mathbb{Z}}^*_{\mathbb{A}}\mathbb{Q}^*, \tau^{\beta}_{\mathbb{A}}\big) = L^2\Big(\mathbb{R}^+, x^{\beta}\frac{d^*x}{\zeta_{\mathbb{A}}(\beta)}\Big)$$

makes perfect sense. For $\beta > 2$, $\phi = \phi_{\mathbb{Z}_{\mathbb{A}}}$ is a cyclic vector, and the space $H^{\beta}_{\mathbb{A}/\mathbb{Q}^*}$ is the completion of $C_c^{\infty}(\mathbb{R}^+)$ with respect to the inner product

$$(f_1, f_2) := \int_0^{\infty} \frac{d^*x|x|^{\beta}}{\zeta_{\mathbb{A}}(\beta)} \sum_{q_1 \in \mathbb{Q}^*} \pi^{\beta}_{\eta}(f_1)\phi(q_1 x)\overline{\sum_{q_2 \in \mathbb{Q}^*} \pi^{\beta}_{\eta}(f_2)\phi(q_2 x)}.$$

We obtain the spectral decomposition of the representation $\pi^{\beta}_{\mathbb{A}/\mathbb{Q}^*}$ of $\mathbb{R}^+$ on $H^{\beta}_{\mathbb{A}/\mathbb{Q}^*}$:

$$H^{\beta}_{\mathbb{A}/\mathbb{Q}^*} \xrightarrow{\sim} L^2\Big(i\mathbb{R}, \zeta_{\mathbb{A}}\big(\frac{\beta}{2}+s, \frac{\beta}{2}-s\big)ds\Big)$$

In the "critical stript", $\beta \in [0,1]$, $\tau^{\beta}_{\mathbb{A}}$ is supported on the hole of $\mathbb{A}$ (e.g., at $\beta = 1$, $\tau^1_{\mathbb{A}}$ is additive Haar measure). The action of $\mathbb{Q}^*$ on $\mathbb{A}$ being ergodic we need to work with the non-commutative algebra $\mathcal{S}(\mathbb{A}) \rtimes \mathbb{Q}^*$ instead of $\mathcal{S}(\mathbb{A}/\mathbb{Q}^*) = \big\{\sum_{q\in\mathbb{Q}^*} \varphi(qx) \,\big|\, \varphi \in \mathcal{S}(\mathbb{A}), x \in \mathbb{A}^*/\mathbb{Q}^*\big\}$.

Next we look at the global semi-group. We have the function,

$$G^{\beta}_{\mathbb{A}}(x) := \prod_{p \geq \eta} G^{\beta}_p(x_p) \qquad (x \in \mathbb{A}).$$

Then $G^{\beta}_{\mathbb{A}}(x)$ is well-defined for $\mathrm{Re}(\beta) > 1$ and is supported at $[-1,1] \times \widehat{\mathbb{Z}} \subseteq \mathbb{A}$ and is $\widehat{\mathbb{Z}}^*_{\mathbb{A}}\{\pm 1\}$-invariant. We also define

$$G^{\beta}_{\mathbb{Q}}(x) := \sum_{q \in \mathbb{Q}^*/\{\pm 1\}} G^{\beta}_{\mathbb{A}}(qx) \quad \text{for} \quad x \in \mathbb{A}^*/\widehat{\mathbb{Z}}^*_{\mathbb{A}}\mathbb{Q}^* = \mathbb{R}^+.$$

Then we have

$$G^{\beta}_{\mathbb{Q}}(x) = \zeta_{\mathbb{A}}(\beta) \sum_{1 \leq n \leq 1/x} \frac{1}{\prod_{\eta, p|n} \zeta_p(\beta)} (1 - n^2x^2)_+^{\frac{\beta}{2}-1}.$$

Note that $G^{\beta}_{\mathbb{Q}}(x)$ is a finite sum, is supported at $x \in (0,1)$, is well-defined for all β, has poles at $x = 1/n$ for $n \in \mathbb{Z}_{\geq 1}$ and is $O(1/x)$ for $x \downarrow 0$ with $\big|\frac{1}{n} - x\big| > \frac{\varepsilon}{n}$, $\varepsilon > 0$. We have

$$\begin{aligned} G^{\beta}_{\mathbb{Q}}\varphi(y) &= \int_{\mathbb{R}^+} \frac{d^*x|x|^{\beta}_{\eta}}{\zeta_{\mathbb{A}}(\beta)} G^{\beta}_{\mathbb{Q}}(y/x)\varphi(x) \\ &= \int_{\mathbb{R}^+} d^*x \cdot x^{\beta} \sum_{1 \leq n \leq x/y} \frac{1}{\prod_{\eta, p|n} \zeta_p(\beta)} \big(1 - n^2\frac{y^2}{x^2}\big)_+^{\frac{\beta}{2}-1}\varphi(x). \end{aligned}$$

Take $\varphi \in C_c^\infty(\mathbb{R}^+)$. Then $G_{\mathbb{Q}}^\beta \varphi(y)$ converges for $\mathrm{Re}(\beta) > 0$ and can be holomorphically extended for all β. Remark that if $\mathrm{Supp}(\varphi) \subseteq \{|x|_\eta \leq c\}$, then $\mathrm{Supp}(G_{\mathbb{Q}}^\beta \varphi) \subseteq \{|x|_\eta \leq c\}$. We have

$$G_{\mathbb{Q}}^{\beta_1} G_{\mathbb{Q}}^{\beta_2} = G_{\mathbb{Q}}^{\beta_1+\beta_2} \qquad (\mathrm{Re}(\beta_i) > 0,\ i = 1, 2),$$
$$\lim_{\beta \to 0} G_{\mathbb{A}}^\beta = \mathrm{id}.$$

Let us look at the Mellin transform of $G_{\mathbb{Q}}^\beta$;

$$\int_0^1 d^*x \cdot x^s G_{\mathbb{Q}}^\beta(x) = \zeta_{\mathbb{A}}(\beta, s) \qquad (\mathrm{Re}(s) > 1, \mathrm{Re}(\beta) > 0). \tag{6.6}$$

In fact, it can be calculated as

$$\begin{aligned}\int_0^1 d^*x \cdot x^s G_{\mathbb{Q}}^\beta(x) &= \int_0^1 d^*x \cdot x^s \sum_{1 \leq n \leq 1/x} \frac{\zeta_{\mathbb{A}}(\beta)}{\prod_{\eta, p|n} \zeta_p(\beta)} (1 - n^2x^2)^{\frac{\beta}{2}-1} \\ &= \zeta_{\mathbb{A}}(\beta) \sum_{n \geq 1} \frac{1}{\prod_{p|n} \zeta_p(\beta)} \frac{1}{\zeta_\eta(\beta)} \int_0^{1/n} d^*x \cdot x^s (1 - n^2x^2)^{\frac{\beta}{2}-1} \\ &= \zeta_{\mathbb{A}}(\beta) \Big[\sum_{n \geq 1} \big(\prod_{p|n} (1 - p^{-\beta})\big) \frac{1}{n^s}\Big] \frac{1}{\zeta_\eta(\beta)} \int_0^1 d^*x \cdot x^s (1 - x^2)^{\frac{\beta}{2}-1}.\end{aligned}$$

Here we have

$$\begin{aligned}\sum_{n \geq 1} \big(\prod_{p|n} (1 - p^{-\beta})\big) \frac{1}{n^s} &= \prod_{p > \eta} \Big(1 + (1 - p^{-\beta}) \sum_{k \geq 1} p^{-ks}\Big) \\ &= \prod_{p > \eta} \Big(1 + \frac{(1 - p^{-s})p^{-s}}{1 - p^{-s}}\Big) = \prod_{p > \eta} \frac{1 - p^{-\beta-s}}{1 - p^{-s}} = \prod_{p > \eta} \frac{\zeta_p(s)}{\zeta_p(\beta + s)}\end{aligned}$$

and

$$\int_0^1 d^*x \cdot x^s (1 - x^2)^{\frac{\beta}{2}-1} = \frac{\zeta_\eta(\beta)\zeta_\eta(s)}{\zeta_\eta(\beta + s)}.$$

Therefore we obtain

$$\begin{aligned}\int_0^1 d^*x \cdot x^s G_{\mathbb{Q}}^\beta(x) &= \zeta_{\mathbb{A}}(\beta) \cdot \prod_{p > \eta} \frac{\zeta_p(s)}{\zeta_p(\beta + s)} \cdot \frac{1}{\zeta_\eta(\beta)} \cdot \frac{\zeta_\eta(\beta)\zeta_\eta(s)}{\zeta_\eta(\beta + s)} \\ &= \frac{\zeta_{\mathbb{A}}(\beta)\zeta_{\mathbb{A}}(s)}{\zeta_{\mathbb{A}}(\beta + s)} \\ &= \zeta_{\mathbb{A}}(\beta, s),\end{aligned}$$

whence the global formula (6.6).

Notice that (6.6) can be written as

$$\int_0^1 d^*x \cdot x^s \frac{1}{\zeta_{\mathbb{A}}(\beta)} G_{\mathbb{Q}}^\beta(x) = \frac{\zeta_{\mathbb{A}}(s)}{\zeta_{\mathbb{A}}(\beta + s)}.$$

By the Mellin inversion formula, we obtain

$$\frac{1}{\zeta_{\mathbb{A}}(\beta)} G_{\mathbb{Q}}^{\beta}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{\zeta_{\mathbb{A}}(s)}{\zeta_{\mathbb{A}}(\beta+s)} ds \qquad (c > 1). \tag{6.7}$$

Then we obtain the following corollary:

Corollary 6.3.1. *Fix $\beta > 2$ so that $G_{\mathbb{Q}}^{\beta}(x)$ is a continuous function of $x \in (0,1]$. Then the Riemann hypothesis is equivalent to the statement*

$$\frac{1}{\zeta_{\mathbb{A}}(\beta)} G_{\mathbb{Q}}^{\beta}(x) \sim \frac{A_\beta}{x} + A'_\beta + O(x^{\beta - \frac{1}{2} - \varepsilon}) \qquad (x \downarrow 0) \tag{6.8}$$

for all $\varepsilon > 0$. Here the constants A_β and A'_β are given by

$$A_\beta := \operatorname*{Res}_{s=1} \frac{\zeta_{\mathbb{A}}(s)}{\zeta_{\mathbb{A}}(\beta+s)} = \frac{1}{\zeta_{\mathbb{A}}(1+\beta)}, \qquad A'_\beta := \operatorname*{Res}_{s=0} \frac{\zeta_{\mathbb{A}}(s)}{\zeta_{\mathbb{A}}(\beta+s)} = -\frac{1}{\zeta_{\mathbb{A}}(\beta)},$$

In fact, if we have the Riemann hypothesis, we know all the poles of the function $\zeta_{\mathbb{A}}(s)/\zeta_{\mathbb{A}}(\beta+s)$ are at $s = 1, 0$ and at $\mathrm{Re}(s) = \frac{1}{2} - \beta$. Then we can shift the integral in (6.7) to $c = \frac{1}{2} - \beta + \varepsilon$, $\varepsilon > 0$ and pick up the residue at $s = 1, 0$ obtaining the estimate (6.8). Conversely if we have such an estimate, (6.6) shows that $\zeta_{\mathbb{A}}(s)/\zeta_{\mathbb{A}}(\beta+s)$ is holomorphic in $\mathrm{Re}(s) > \frac{1}{2} - \beta + \varepsilon$, for any $\varepsilon > 0$, except for the simple poles at $s = 1, 0$.

Remark that, for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(\beta) > 0$, we have

$$\begin{aligned}
(G_{\mathbb{Q}}^{\beta}\varphi)^{\wedge}(s) &= \int_{\mathbb{R}^+} d^*y \cdot y^s G_{\mathbb{Q}}^{\beta}\varphi(y) \\
&= \int_{\mathbb{R}^+} d^*y \cdot y^s \int_0^1 d^*x \cdot x^\beta \varphi(x) \sum_{n \le x/y} \frac{1}{\prod_{\eta, p|n} \zeta_p(\beta)} (1 - n^2y^2/x^2)_+^{\frac{\beta}{2}-1} \\
&= \int_0^1 d^*x \cdot x^{\beta+s} \varphi(x) \int_0^1 d^*y \cdot y^s \sum_{n \le 1/y} \frac{1}{\prod_{\eta, p|n} \zeta_p(\beta)} (1 - n^2y^2)_+^{\frac{\beta}{2}-1} \\
&= \widehat{\varphi}(s+\beta) \frac{\zeta_{\mathbb{A}}(s)}{\zeta_{\mathbb{A}}(\beta+s)}.
\end{aligned}$$

Hence we obtain diagram

$$\begin{array}{ccc}
L^2(\mathbb{R}^+, d^*x) & \xrightarrow{\ \sim\ } & L^2\left(i\mathbb{R}, \frac{ds}{2\pi i}\right) \\
\circlearrowleft & & \circlearrowleft \\
G_{\mathbb{Q}}^{\beta} & & \widehat{G}_{\mathbb{Q}}^{\beta}
\end{array}$$

where $\widehat{G}_{\mathbb{Q}}^{\beta} : L^2\left(i\mathbb{R}, \frac{ds}{2\pi i}\right) \to L^2\left(i\mathbb{R}, \frac{ds}{2\pi i}\right)$, which corresponds to $G_{\mathbb{Q}}^{\beta}$, is explicitly given by

$$\widehat{G}_{\mathbb{Q}}^{\beta} \widehat{f}(s) := \frac{\zeta_{\mathbb{A}}(s)}{\zeta_{\mathbb{A}}(\beta+s)} \widehat{f}(s+\beta) = \zeta_{\mathbb{A}}(s) \cdot t_\beta \cdot \frac{1}{\zeta_{\mathbb{A}}(s)} \cdot \widehat{f}(s),$$

and t_β is the translation by β. It makes sense for $\widehat{f}(s)$ holomorphic in $\mathrm{Re}(s) \geq 0$, and $\mathrm{Re}(\beta) \geq 0$, or for general $\widehat{f}(s)$ for $\beta \in i\mathbb{R}$. The semi-group property of $G_{\mathbb{Q}}^{\beta}$ follows since t_β is a semi-group. Therefore it makes sense to calculate the infinitesimal generator:

$$\frac{\partial}{\partial \beta}\Big|_{\beta=0} G_{\mathbb{Q}}^{\beta} \widehat{f}(s) = \zeta_{\mathbb{A}}(s) \cdot \frac{\partial}{\partial s} \frac{1}{\zeta_{\mathbb{A}}(s)} \cdot \widehat{f}(s) = \frac{\partial}{\partial s} \widehat{f}(s) - \widehat{f}(s) d \log \zeta_{\mathbb{A}}(s)$$

Notice that, in the explicit sum formula, we can write

$$\sum_{\zeta_{\mathbb{A}}(s)=0} \widehat{f}(s) - \widehat{f}(0) - \widehat{f}(1) = \frac{1}{2\pi i} \oint \Big[\frac{\partial}{\partial s} \widehat{f}(s) - \widehat{f}(s) d \log \zeta_{\mathbb{A}}(s) \Big],$$

the path taken around all the zeros and poles of the global zeta function $\zeta_{\mathbb{A}}(s)$; $\frac{\partial}{\partial s} \widehat{f}(s)$ does not contribute to the integral being holomorphic. The infinitesimal generator of this semi-group is hence closely related to the Riemann hypothesis.

7

Higher Dimensional Theory

Summary. In Chap. 7 we study the representation $GL_{d+1}(\mathbb{Z}_p) \to U(H)$ where $H = L^2(\mathbb{P}^d(\mathbb{Z}_p))$. The commutant of $GL_{d+1}(\mathbb{Z}_p)$ is generated by the functions on $\Omega_d := B_{d,1}\backslash GL_{d+1}/B_{d,1}$. Note that the measure on the space Ω_d is induced from the Haar measure on $GL_{d+1}(\mathbb{Z}_p)$ and is given by the β-measure with the appropriate parameters. Ω_d parameterizes the relative position of two lines in $\mathbb{P}^d(\mathbb{Z}_p)$ which is given by the "angle" between them, a real number in the real case and an integer in the p-adic cases.

Remember that for real prime η

$$GL_{d+1}(\mathbb{Z}_\eta) = \begin{cases} O_{d+1} & \text{if } \eta \text{ is real,} \\ U_{d+1} & \text{if } \eta \text{ is complex,} \end{cases} \qquad B_{d,1}(\mathbb{Z}_\eta) = \begin{cases} O_d \times O_1 & \text{if } \eta \text{ is real,} \\ U_d \times U_1 & \text{if } \eta \text{ is complex.} \end{cases}$$

Here O_d is the orthogonal group and U_d is the unitary group of size $d \times d$. Then, in the real case, we get a finite angle $\sin\theta$ between two lines. In the p-adic case, we get an integer N. Hence we have

$$\Omega_1 = B_{1,1}\backslash GL_2/B_{1,1} = \begin{cases} p^{\mathbb{N}} \cup \{0\} \simeq p^{-\mathbb{N}} \cup \{0\} & p \text{ is finite,} \\ |\sin\theta| & \text{if } \eta \text{ is real,} \end{cases} \subset [0,1].$$

We have also idempotents e_n in the Hecke algebra $\mathcal{H}$, and H is decomposed as

$$H = \bigoplus_{n \geq 0} H * e_n.$$

In our cases, we see that these idempotents are given by the Jacobi functions with certain parameter both in the p-adic and in the real cases. In the case of the p-adic, we note that the Hecke algebra $C^\infty(\Omega_d)$ grows by one dimension every step from module p^N to modulo p^{N+1}. There is one new representation. Also our basis $\varphi^{\alpha,\beta}_{p(N)}$ (this also factor through modulo p^N) are defined to be orthogonal to every basis counted before together with the normalization

$$\varphi^{\alpha,\beta}_{p(N)}(0) = ||\varphi^{\alpha,\beta}_{p(N)}||^2.$$

This gives the desired identification of the idempotents with the Jacobi basis.

In the real case, we have the Laplacian $\Delta^{\alpha,\beta}$. Since it commutes with the representation, all idempotents are eigenfunctions. The Laplacian is identified up to a constant multiple with $\Delta^{\alpha,\beta} = DD^{+}$ and thus the idempotents are again identified with the Jacobi basis.

7.1 Higher Dimensional Cases

7.1.1 *q*-*β*-Chain

In this chapter we introduce the higher dimensional (rank 1) theory. We consider the projective r-space

$$B_{1,\dots,1}\backslash\mathbb{P}^{r-1} = B_{1,\dots,1}\backslash GL_r/B_{1,r-1}$$

where

$$B_{1,\dots,1} = \left\{ \begin{pmatrix} * & \cdots & \cdots & * \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix} \in GL_r \right\},$$

$$B_{1,r-1} = \left\{ \left(\begin{array}{c|ccc} 1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right) \in GL_r \,\middle|\, A \in GL_{r-1} \right\}.$$

Here $B_{1,\dots,1}$ is the minimal parabolic subgroup of GL_r and $B_{1,r-1}$ is the maximal. Let us first construct the q-theory and then take the p-adic and real limit, respectively.

Now take the state space $\mathbb{N}^r = \bigsqcup_{N\geq 0} X_N$ where

$$X_N := \left\{\overline{m} \in \mathbb{N}^r \,\middle|\, |\overline{m}| := m_1 + \cdots + m_r = N\right\}, \qquad \#X_N = \binom{N+r-1}{N}.$$

The probability $P^{\overline{\alpha}}_{q(N)}$ on X_N is given as follows; Take a sequence of parameter $\overline{\alpha} := (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r_{>0}$. Then $P^{\overline{\alpha}}_{q(N)}$ is defined as

$$P^{\overline{\alpha}}_{q(N)}(\overline{m},\overline{n}) := \begin{cases} \dfrac{1-q^{\alpha_i+m_i}}{1-q^{|\overline{\alpha}|+|\overline{m}|}} q^{\sum_{j<i}(\alpha_j+m_j)} & \text{if } \overline{n} = \overline{m}(i+) \text{ for some } 1 \leq i \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Here we denote $\overline{m}(i+) := (m_1, \dots, m_{i-1}, m_i + 1, m_{i+1}, \dots, m_r)$. For example at the origin $\overline{m} = 0$, the probability is given as follows;

$$P^{\overline{\alpha}}_{q(N)}\big(\overline{0},(1,0,0,0,\dots,0)\big) = \frac{1-q^{\alpha_1}}{1-q^{|\alpha|}},$$
$$P^{\overline{\alpha}}_{q(N)}\big(\overline{0},(0,1,0,0,\dots,0)\big) = \frac{(1-q^{\alpha_2})q^{\alpha_1}}{1-q^{|\alpha|}},$$
$$P^{\overline{\alpha}}_{q(N)}\big(\overline{0},(0,0,1,0,\dots,0)\big) = \frac{(1-q^{\alpha_3})q^{\alpha_1+\alpha_2}}{1-q^{|\alpha|}}, \qquad \dots .$$

At the point $\overline{m}$, we replace $\overline{\alpha}$ with $\overline{\alpha}+\overline{m}$. The probability measure $\tau^{\overline{\alpha}}_{q(N)}$ on the N-th layer X_N is given as follows;

$$\tau^{\overline{\alpha}}_{q(N)} := (P^{\overline{\alpha}*}_{q(N)})^N \delta_{\overline{0}} = (A_1 + \cdots + A_r)^N \delta_{\overline{0}},$$

where

$$A_i\varphi(\overline{m}) := \frac{1-q^{\alpha_i+m_i-1}}{1-q^{|\overline{\alpha}|+|\overline{m}|-1}} q^{\sum_{j<i}(\alpha_j+m_j)} \varphi(\overline{m}(i-))$$

Now again the operator A_j satisfies the q-commutativity:

$$A_iA_j = qA_jA_i \qquad \text{for } j<i.$$

Therefore we can apply the higher dimensional q-binomial theorem to calculate $\tau^{\overline{\alpha}}_{q(N)}$ and obtain the formula

$$\tau^{\overline{\alpha}}_{q(N)}(\overline{m}) = \begin{bmatrix} N \\ \overline{m} \end{bmatrix}_q \frac{\zeta_q(\overline{\alpha}+\overline{m})}{\zeta_q(\overline{\alpha})} q^{\sum_{j<i}\alpha_j m_i}$$

where $\left[\begin{smallmatrix} N \\ \overline{m} \end{smallmatrix}\right]_q$ is the multi-variable q-binomial coefficient and $\zeta_q(\overline{\alpha})$ is the multi-variable beta function defined respectively by

$$\begin{bmatrix} N \\ \overline{m} \end{bmatrix}_q := \frac{[N]_q!}{[m_1]_q!\cdots[m_r]_q!}, \qquad (m_1+\cdots+m_r = N),$$
$$\zeta_q(\overline{\alpha}) := \frac{\zeta_q(\alpha_1)\cdots\zeta_q(\alpha_r)}{\zeta_q(\alpha_1+\cdots+\alpha_r)\zeta_q(1)^{r-1}}.$$

Hence once we know these, it is easy to see that the process with $\overline{\alpha}$ starting at $\overline{m}$ is the same as the process with $\overline{\alpha}+\overline{m}$ starting from 0. Therefore, the Green kernel $G^{\overline{\alpha}}$ is given by

$$G^{\overline{\alpha}}(\overline{m},\overline{n}) = \begin{cases} \tau^{\overline{\alpha}+\overline{m}}_{q(N)}(\overline{n}-\overline{m}) & \text{if } \overline{n} \geq \overline{m}, \\ 0 & \text{otherwise}. \end{cases}.$$

and the Martin kernel is

$$K^{\overline{\alpha}}(\overline{m},\overline{n}) = \begin{cases} \dfrac{\tau^{\overline{\alpha}+\overline{m}}_{q(N)}(\overline{n}-\overline{m})}{\tau^{\overline{\alpha}}_{q(N)}(\overline{n})} & \text{if } \overline{n} \geq \overline{m}, \\ 0 & \text{otherwise}. \end{cases}$$

Here $\overline{n} \geq \overline{m}$ means that $n_j \geq m_j$ for all j.

Then the boundary space ∂X of the q-processes is calculated as

$$\partial X = \mathbb{N}^{r-1} \sqcup \mathbb{N}^{r-2} \sqcup \cdots \sqcup \mathbb{N}^1 \sqcup \mathbb{N}^0,$$

where $\mathbb{N}^0 = \{\infty\}$ and the element of $\mathbb{N}^{r-j}$ can be expressed as

$$(\underbrace{\infty, \dots, \infty}_{j}, m_{j+1}, \dots, m_r) = \lim_{m_1, \dots, m_j \to \infty} (m_1, \dots, m_j, m_{j+1}, \dots, m_r).$$

Note that the set $\mathbb{N}^{r-1}$ is the main part of the boundary ∂X which is open and dense subset. The extension of the Martin kernel to the boundary is given as

$$K^{\overline{\alpha}}(\overline{m}, (\infty, n_2, \dots, n_r)) =$$
$$\begin{cases} \dfrac{\zeta_q(1+n_2)}{\zeta_q(1+n_2-m_2)} \cdots \dfrac{\zeta_q(1+n_r)}{\zeta_q(1+n_r-m_r)} \dfrac{\zeta_q(\overline{\alpha})}{\zeta_q(\overline{\alpha}+\overline{m})} q^{\sum_{j<i}\left(m_j n_i - (m_j+\alpha_j) m_i\right)} & \text{if } n_i \geq m_i, \\ 0 & \text{otherwise.} \end{cases}$$

The harmonic measure $\tau_q^{\overline{\alpha}}$ on ∂X is given by

$$\tau_q^{\overline{\alpha}}(\infty, n_2, \dots, n_r) = \frac{1}{\zeta_q(\overline{\alpha})} \frac{\zeta_q(\alpha_2+n_2)}{\zeta_q(1+n_2)} \cdots \frac{\zeta_q(\alpha_r+n_r)}{\zeta_q(1+n_r)} q^{\sum_{j<i} \alpha_j n_i}.$$

Next we want to obtain the basis and the ladder. We denote by $H_{q(N)}^{\overline{a}} := \ell^2\left(X_N, \tau_{q(N)}^{\overline{\alpha}}\right)$ the Hilbert space of the N-th layer with parameter $\overline{\alpha}$. Consider the following ladder

$$H_{q(N)}^{\overline{a}} \underset{D_N^+}{\overset{D_N}{\rightleftarrows}} H_{q(N-1)}^{\overline{a}(1+,2+)} \underset{D_{N-1}^+}{\overset{D_{N-1}}{\rightleftarrows}} H_{q(N-2)}^{\overline{a}(1++,2++)}$$

where $D_N : H_{q(N)}^{\overline{a}} \to H_{q(N)}^{\overline{a}(1+,2+)}$ is the difference operator given by

$$D_N \varphi(\overline{m}) := q^{-(m_2+\cdots+m_r)} \frac{\varphi(\overline{m}(2+)) - \varphi(\overline{m}(1+))}{1-q}$$

and D_N^+ is the normalization of the adjoint operator D_N^* of D_N defined as follows:

$$D_N^+ := (1-q)^2 \frac{[|\overline{\alpha}|+N]_q [N]_q [\alpha_1]_q [\alpha_2]_q}{[|\overline{\alpha}|]_q [|\overline{\alpha}|+1]_q} D_N^*.$$

Then we have the Heisenberg relation up the ladder

$$D_N D_N^+ - D_{N-1}^+ D_{N-1} = q^{-\alpha_1} [\alpha_1 + \alpha_2]_q \mathrm{id}_{H_{q(N-1)}^{\overline{\alpha}(1+,2+)}}.$$

The reason we normalize the adjoint operator D_N^* is to keep the Heisenberg relation much simpler one (if we do not introduce the operator D^+, the relation above becomes of the form $D_N D_N^* - (\text{const.}) D_{N-1}^* D_{N-1} = \cdots$). Consider the following injection $\rho_{r-1} : H_{q(N)}^{(\alpha_1+\alpha_2,\alpha_3,\dots,\alpha_r)} \hookrightarrow H_{q(N)}^{\overline{\alpha}}$ between $(r-1)$-dimension space and r-dimension space;

$$\rho_{r-1}\varphi(\overline{m}) := \varphi(m_1 + m_2, m_3, \dots, m_r).$$

The adjoint ρ_{r-1}^* of ρ_{r-1} is given by

$$\rho_{r-1}^*\varphi(\overline{m}) = \sum_{i+j=m_1} \varphi(i, j, m_2, m_3, \dots, m_r)\tau_{q(m_1)}^{(\alpha_1)\alpha_2}(i, j).$$

Then one can see that the relation $\operatorname{Image}\rho_{r-1} = \ker D_N$ holds. Hence, using the Heisenberg relation, we obtain the orthogonal basis

$$\varphi_{q(N),\overline{v}}^{\overline{\alpha}} := \frac{(-1)^{|v|} q^{\binom{|\overline{v}|}{2}}}{[v_1]_q! \cdots [v_{r-1}]_q!} (D^+)^{v_1} \rho_{r-1} (D^+)^{v_2} \rho_{r-2} \cdots (D^+)^{v_{r-2}} \rho_2 (D^+)^{v_{r-1}} \mathbf{1}$$

where $\overline{v} = (v_1, \dots, v_{r-1})$ and $|\overline{v}| \leq N$.

Now consider the boundary space. Let $H_q^{\overline{\alpha}} := \ell^2(\mathbb{N}^{r-1}, \tau_q^{\overline{\alpha}})$ be the Hilbert space of the boundary (notice that $\mathbb{N}^{r-1}$ is dense in ∂X). Then we have also the ladder

$$H_q^{\overline{\alpha}} \underset{D^+}{\overset{D}{\rightleftarrows}} H_q^{\overline{\alpha}(1+,2+)} \underset{D^+}{\overset{D}{\rightleftarrows}} H_q^{\overline{\alpha}(1++,2++)}$$

where $D : H_q^{\overline{\alpha}} \to H_q^{\overline{\alpha}(1+,2+)}$ is a difference operator given by

$$D\varphi(\overline{n}) := q^{-|\overline{n}|} \frac{\varphi(\overline{n}(2+)) - \varphi(\overline{n})}{1-q}, \quad \overline{n} = (\infty, n_2, \dots, n_r)$$

and D^+ is adjoint of D up to a constant. They again satisfy the following Heisenberg relation

$$DD^+ - D^+D = q^{-\alpha_1}[\alpha_1 + \alpha_2]_q \mathrm{id}_{\mathcal{H}_q^{\overline{\alpha}(1+,2+)}}$$

We have also the embedding $\rho_{r-1} : H_q^{(\alpha_1+\alpha_2,\alpha_3,\dots,\alpha_r)} \hookrightarrow H_q^{\overline{\alpha}}$ defined by

$$\rho_{r-1}\varphi(\infty, n_2, \dots, n_r) = \varphi(\infty, n_3, \dots, n_r)$$

and see that $\operatorname{Image}\rho_{r-1} = \ker D$. This gives also the orthogonal basis $\varphi_{q,\overline{v}}^{\overline{\alpha}}$ for the boundary;

$$\varphi_{q,\overline{v}}^{\overline{\alpha}} := \frac{(-1)^{|v|} q^{\binom{|\overline{v}|}{2}}}{[v_1]_q! \cdots [v_{r-1}]_q!} (D^+)^{v_1} \rho_{r-1} (D^+)^{v_2} \rho_{r-2} \cdots (D^+)^{v_{r-2}} \rho_2 (D^+)^{v_{r-1}} \mathbf{1}$$

Similarly to the case of $r = 2$, we obtain the compatibility with respect to the Martin kernel between the operators D, D^+, ρ and the corresponding operators on the finite layers. Then we have also the commutative diagram and the diagonalization of the Martin kernel, cf. (4.9).

7.1.2 The p-Adic Limit of the q-β-Chain

Let us take the p-adic limit Ⓟ of the above q-β-chain of the higher dimension. Remember that the p-adic limit Ⓟ is the limit $q \to 0$ with $q^{\alpha_i} \to p^{-\alpha_i}$. We identify $X_N = X_{p(N)} = \left\{\overline{m} \in \mathbb{N}^r \,\middle|\, |\overline{m}| = N\right\}$ with the space

$$B_{1,\dots,1}(\mathbb{Z}/p^N)\backslash GL_r(\mathbb{Z}/p^N)/B_{1,r-1}(\mathbb{Z}/p^N) = B_{1,\dots,1}(\mathbb{Z}/p^N)\backslash\mathbb{P}^{r-1}(\mathbb{Z}/p^N)$$

as follows;

$$\begin{aligned} X_{p(N)} &\xrightarrow{\sim} B_{1,\dots,1}(\mathbb{Z}/p^N)\backslash\mathbb{P}^{r-1}(\mathbb{Z}/p^N) \\ \overline{m} &\longmapsto (1 : p^{m_r} : p^{m_r+m_{r-1}} : \cdots : p^{m_r+\cdots+m_2}). \end{aligned}$$

Then the p-adic limit of the q-β-chain is the tree chain induced from the natural projection $\mathbb{Z}/p^{N+1} \to \mathbb{Z}/p^N$. We also identify the boundary $\partial X = \mathbb{N}^{r-1} \sqcup \mathbb{N}^{r-2} \sqcup \cdots \sqcup \mathbb{N}^1 \sqcup \mathbb{N}^0$ with

$$\begin{aligned} \partial X &\xrightarrow{\sim} B_{1,\dots,1}(\mathbb{Z}_p)\backslash\mathbb{P}^{r-1}(\mathbb{Z}_p) \\ (\infty, m_2, \dots, m_r) &\longmapsto (1 : p^{m_r} : p^{m_r+m_{r-1}} : \cdots : p^{m_r+\cdots+m_2}) \end{aligned}$$

Here we interpret $p^\infty = 0$. The harmonic measure $\tau_p^{\overline{\alpha}}$ on $B_{1,\dots,1}(\mathbb{Z}_p)\backslash\mathbb{P}^{r-1}(\mathbb{Z}_p)$ is also given by the projection of the product of the p-adic γ measures;

$$\tau_p^{\overline{\alpha}} = \mathrm{pr}_*(\tau_{\mathbb{Z}_p}^{\alpha_1} \otimes \cdots \oplus \tau_{\mathbb{Z}_p}^{\alpha_r}).$$

The basis $\varphi_{q(N),\overline{v}}^{\overline{\alpha}}$ and $\varphi_{q,\overline{v}}^{\overline{\alpha}}$ converge in the p-adic limit to the basis $\varphi_{p(N),\overline{v}}^{\overline{\alpha}}$ and $\varphi_{p,\overline{v}}^{\overline{\alpha}}$ for the finite layers spaces $\ell^2(X_{p(N)}, \tau_p^{\overline{\alpha}})$, respectively.

7.1.3 The Real Limit of the q-β-Chain

We next treat the real limit Ⓨ. First of all, the measure $\tau_{\eta(N)}^{\overline{\alpha}}$ on the finite layer X_N is given by

$$\tau_{\eta(N)}^{\overline{\alpha}}(\overline{m}) := \binom{N}{\overline{m}} \frac{\zeta_\eta(\overline{\alpha} + 2\overline{m})}{\zeta_\eta(\overline{\alpha})},$$

where $\binom{N}{\overline{m}}$ is the multi-variable binomial coefficient and $\zeta_\eta(\overline{\alpha})$ is the multi-variable beta function which are respectively given by

$$\begin{aligned} \binom{N}{\overline{m}} &:= \frac{N!}{m_1! \cdots m_r!} \qquad (m_1 + \cdots + m_r = N), \\ \zeta_\eta(\overline{\alpha}) &:= \frac{\zeta_\eta(\alpha_1) \cdots \zeta_\eta(\alpha_r)}{\zeta_\eta(\alpha_1 + \cdots + \alpha_r)}. \end{aligned}$$

Now the sequence $\{m^{(k)}\} = \{(m_1^{(k)}, \dots, m_r^{(k)})\}$ converges with respect to the Martin metric if and only if there exists some x_j such that

$$x_j^2 = \lim_{k\to\infty} \frac{m_j^{(k)}}{|m^{(k)}|} \in [0,1] \quad \text{for all } j = 1, \dots, r.$$

Note that $x_1^2 + \cdots + x_r^2 = 1$. Hence we identify the boundary ∂X with

$$\{\overline{x} \in [0,1]^r \mid x_1^2 + \cdots + x_r^2 = 1\} = \{\pm 1\}^r \backslash S^{r-1} = \begin{cases} \{\pm 1\}^{r-1} \backslash \mathbb{P}^{r-1}(\mathbb{R}) & \text{if } \eta \text{ is real,} \\ (\mathbb{C}^{(1)})^{r-1} \backslash \mathbb{P}^{r-1}(\mathbb{C}) & \text{if } \eta \text{ is complex.} \end{cases}$$

Then the harmonic measure $\tau_\eta^{\overline{\alpha}}$ is given by the β-measure;

$$\tau_\eta^{\overline{\alpha}}(\overline{x}) = \mathrm{pr}_*(\tau_{\mathbb{Z}_\eta}^{\alpha_1} \otimes \cdots \oplus \tau_{\mathbb{Z}_\eta}^{\alpha_r}) = |x_1|^{\alpha_1 - 1} \cdots |x_r|^{\alpha_r - 1} \frac{d^0\overline{x}}{\zeta_\eta(\overline{\alpha})}.$$

Here $d^0\overline{x}$ is the Haar measure on the $(r-1)$-sphere S^{r-1} normalized by $d^0\overline{x}(S^{r-1}) = 1$.

Let $H_\eta^{\overline{\alpha}} := L^2\big(\{\pm 1\}^r \backslash S^{r-1}, \tau_\eta^{\overline{\alpha}}\big)$ the Hilbert space of the boundary. Then the operator $D : H_\eta^{\overline{\alpha}} \to H_\eta^{\overline{\alpha}(1++,2++)}$ and its adjoint D^+ (up to a constant multiple) converge to certain differential operators as in the 2-dimensional case. Then we get the ladder

$$H_\eta^{\overline{\alpha}} \underset{D^+}{\overset{D}{\rightleftarrows}} H_\eta^{\overline{\alpha}(1++,2++)} \underset{D^+}{\overset{D}{\rightleftarrows}} H_\eta^{\overline{\alpha}(1++++,2++++)}$$

We have also the injection $\rho_{r-1} : H_\eta^{(\alpha_1+\alpha_2,\alpha_3,\dots,\alpha_r)} \hookrightarrow H_\eta^{\overline{\alpha}}$, it is given as follows;

$$\rho_{r-1}\varphi(\overline{x}) = \varphi\big(\sqrt{x_1^2 + x_2^2}, x_3, \dots, x_r\big).$$

Here we replace $x_1^2 + x_2^2$ with $|x_1|_\eta^2 + |x_2|_\eta^2$ if η is complex. Since $\mathrm{Image}\,\rho_{r-1} = \ker D$, we get again the basis as before denoted by $\varphi_{\eta,\overline{v}}^{\overline{\alpha}}$, which can be written as the limit $N \to \infty$ of the basis $\varphi_{\eta(N),\overline{v}}^{\overline{\alpha}}$ of the finite layer or the limit $q \to 1$ of the basis $\varphi_{q,\overline{v}}^{\overline{\alpha}}$ of the boundary space for the q-β-chain. Also one can obtain the commutative diagram generalizing (4.9).

Notice also that if $\overline{\alpha} = (1, \dots, 1)$, the harmonic measure $\tau_\eta^{(1,\dots,1)}$ is the unique probability measure on the projective r-space $\mathbb{P}^{r-1}$ which is invariant under the action of $GL_r(\mathbb{Z}_p)$ in the p-β-chain, or O_r, in the real β-chain, or U_r in the complex β-chain.

7.2 Representations of $GL_d(\mathbb{Z}_p)$, $p \geq \eta$, on Rank-1 Symmetric Spaces

We next concentrate on the q-Jacobi basis, which is the q-interpolation of zonal spherical functions. So let us begin with general setting.

Let G be a compact group and τ the Haar measure on G normalized by $\tau(G) = 1$. Take a closed subgroup B of G and we let $X := G/B$. We have a natural projection $\mathrm{pr} : G \twoheadrightarrow X$. Let $\tau_X = \mathrm{pr}_* \tau$, a probability measure on X. Let $H = L^2(X, \tau_X)$ and π the unitary representation of G defined by

$$\pi : G \longrightarrow U(H), \qquad \pi(g)f(x) = f(g^{-1}x).$$

Since G is compact, we have the following irreducible decompositions

$$H = \bigoplus_m V_m, \qquad \pi = \bigoplus_m \pi_m \tag{7.1}$$

and V_m are finite dimensional. How this representation decomposes can be seen by looking at the space $\Omega := G\backslash X \times X \simeq B\backslash X/B$. Here the isomorphism is given by

$$\begin{aligned} G\backslash X \times X &\xrightarrow{\sim} B\backslash X/B, \\ G(g_1B, g_2B) &\longmapsto Bg_1^{-1}g_2B, \\ G(\mathrm{id}B, gB) &\longleftarrow BgB. \end{aligned}$$

We have also the natural projection $\mathrm{pr} : G \twoheadrightarrow \Omega$ and obtain the probability measure $\tau_\Omega := \mathrm{pr}_* \tau$ on Ω.

Let $\mathcal{H} = C(\Omega)$ or $L^1(\Omega)$ or $C^\infty(\Omega)$. Note that in the p-adic case, the smoothness means locally constant. Now we give $\mathcal{H}$ the structure of the convolution algebra. For two functions φ_1 and φ_2 on G, we define the new function $\varphi_1 * \varphi_2$ on G by the convolution product;

$$\varphi_1 * \varphi_2(g) := \int_G \varphi_1(g_1)\varphi_2(g_1^{-1}g)\tau(dg_1).$$

If both φ_1 and φ_2 are invariant under both side by B, then the convolution $\varphi_1 * \varphi_2$ is also invariant under both side. Further we have $f * \varphi \in H$ for any $f \in H$ and $\varphi \in \mathcal{H}$. This is a $*$-representation of $\mathcal{H}$ on the Hilbert space H where the $*$-algebra structure is

$$\varphi^*(g) := \overline{\varphi(g^{-1})}.$$

i.e., $\mathcal{H}$ acts on H on the right by convolution.

$$H \times \mathcal{H} \longrightarrow H, \qquad (f, \varphi) \longmapsto f * \varphi.$$

This action commutes with the G-action, that is, $(\pi(g)f) * \varphi = \pi(g)(f * \varphi)$. Further we see that the commutant of $\pi(G)$ is always generated by $\mathcal{H}$. Let $K : H \to H$ be some operator which commutes with the G-action. Then K can be written as the convolution with some kernel $K : X \times X \to H$ as

$$Kf(y) = \int_X f(x)K(x, y)\tau_X(dx).$$

We use the same symbol for the kernel. Then we have for any $f \in H$ and $g \in G$

$$\pi(g)Kf(y) = \int_X f(x)K(x, g^{-1}y)\tau_X(dx)$$
$$= K(\pi(g)f)(y) = \int_X f(x)K(gx, y)\tau_X(dx).$$

This shows that $K(x, g^{-1}y) = K(gx, y)$, or $K(x, y) = K(gx, gy)$. Therefore the kernel K is a function on $G\backslash X \times X = \Omega$. Hence, if we put $K(x, y) =: k(x^{-1}y)$, we have $k \in \mathcal{H}$ and $Kf = f * k$.

Now assume that we have no multiplicity in the decomposition (7.1). This means that $V_{m_1} \not\simeq V_{m_2}$ for $m_1 \neq m_2$ and also that

$$(\text{the commutant of } \pi(G)) \simeq \bigoplus_m \mathbb{C} \cdot \mathrm{id}_{V_m}$$

by the Schur lemma. Furthermore this is equivalent to that $\mathcal{H}$ is commutative with respect to the convolution product. Hence we can write

$$\mathcal{H} = \bigoplus_m \mathbb{C}e_m,$$

where e_n is an idempotent. Namely, e_n satisfies

$$e_{m_1} * e_{m_2} = \delta_{m_1,m_2}e_{m_1},$$
$$e_m^* = e_m.$$

Therefore $\mathcal{H}$ is very simple algebra once we know the idempotent e_m and the decomposition (7.1) can be easily described. Actually, e_m gives the representation V_m by $V_m = H * e_m$. On the other hand V_m also determines uniquely e_m by the following relation $(V_m)^B = \mathbb{C} \cdot e_m$. These show that V_m determines the e_m up to a constant multiple. Further the constant can be known by the formula

$$||e_m||^2_{\mathcal{H}} = e_m * e_m^*(\mathrm{id}) = e_m(\mathrm{id}).$$

i.e., for any $v \in (V_m)^B$ we have

$$e_m = \frac{v(\mathrm{id})}{||v||^2_{\mathcal{H}}}v.$$

Similarly let us take two closed subgroups B_i $(i = 1, 2)$ of G. Let $H_i := L^2(G/B_i, \tau_{G/B_i})$ and, say, $\mathcal{H}_{B_i} = L^1(B_i\backslash G/B_i, \tau_{B_i\backslash G/B_i})$. If the operator $\varphi : H_1 \to H_2$ commutes with the G-action, that is, $\varphi(\pi(g)f) = \pi(g)(\varphi(f))$ for all $g \in G$, it is given by the convolution

$$\varphi : H_1 \longrightarrow H_2, \qquad f \longmapsto f * \varphi$$

with the kernel φ in $\mathcal{H}_{B_1,B_2} := L^1(B_1\backslash G/B_2, \tau_{B_1\backslash G/B_2})$. Then $\mathcal{H}_{B_1,B_2}$ is a $\mathcal{H}_{B_1}$-module on the left and a $\mathcal{H}_{B_2}$-module on the right. If we assume the multiplicity free of the irreducible decomposition of both H_1 and H_2, then $\mathcal{H}_{B_1,B_2}$ is the module;

$$\mathcal{H}_{B_1,B_2} \simeq \bigoplus_m \mathbb{C}e_m.$$

Here e_m are the common idempotents acting in both the decompositions of H_1 and H_2.

We next renormalize all our old basis. First we redefine the q-Hahn basis as follows;

$$\varphi^{(\alpha)\beta}_{q(N),m} := \frac{\varphi^{(\alpha)\beta}_{q(N),m}(N,0)}{||\varphi^{(\alpha)\beta}_{q(N),m}||^2_{H^{(\alpha)\beta}_{q(N)}}}\varphi^{(\alpha)\beta}_{q(N),m}.$$

Also let us normalize the basis for the finite layer for the p-adic and the real basis in the same way. Similarly on the boundary, we normalize again the q-Jacobi basis;

$$\varphi^{(\alpha)\beta}_{q,m} = \frac{\varphi^{(\alpha)\beta}_{q,m}(0)}{||\varphi^{(\alpha)\beta}_{q,m}||^2_{H^{(\alpha)\beta}_q}}\varphi^{(\alpha)\beta}_{q,m}$$

and similarly for the p-adic and real basis. Here $0 = \lim_{i\to\infty} g^i$. These normalization guarantee the condition $||\varphi^{(\alpha)\beta}_{q,m}||^2 = \varphi^{(\alpha)\beta}_{q,m}(\mathrm{id})$. Each normalized constant can be written as follows; For the q-Hahn basis, it is given by

$$\frac{\varphi^{(\alpha)\beta}_{q(N),m}(N,0)}{||\varphi^{(\alpha)\beta}_{q(N),m}||^2_{H^{(\alpha)\beta}_{q(N)}}} = \frac{\zeta_q(\alpha)}{\zeta_q(\alpha+m)}\frac{\zeta_q(\alpha+\beta+m)}{\zeta_q(\alpha+\beta)}\frac{1-q^{\alpha+\beta+2m-1}}{1-q^{\alpha+\beta+m-1}}.$$

Similarly for the q-Jacobi basis, it is given by the same constant

$$\frac{\varphi^{(\alpha)\beta}_{q,m}(0)}{||\varphi^{(\alpha)\beta}_{q,m}||^2_{H^{(\alpha)\beta}_q}} = \frac{\zeta_q(\alpha)}{\zeta_q(\alpha+m)}\frac{\zeta_q(\alpha+\beta+m)}{\zeta_q(\alpha+\beta)}\frac{1-q^{\alpha+\beta+2m-1}}{1-q^{\alpha+\beta+m-1}}.$$

Notice that when we take $\alpha \to \infty$, both the q-Laguerre and the finite q-Laguerre basis (hence the p-adic and real Laguerre basis as well) are already normalized

$$\frac{\varphi^{\beta}_{q(N),m}(N,0)}{||\varphi^{\beta}_{q(N),m}||^2_{H^{\beta}_{q(N)}}} = \frac{\varphi^{\beta}_{\mathbb{Z}_q,m}(0)}{||\varphi^{\beta}_{\mathbb{Z}_q,m}||^2_{H^{\beta}_{\mathbb{Z}_q}}} = 1.$$

In the following table, we give some examples of the compact group G, the closed subgroup B of G, $X = G/B$, $\Omega = B\backslash G/B$, the probability measure τ_Ω on G and the idempotent e_m which are normalized as above. The first table is for the p-adic cases and the second one for the real cases.

G	B	X	Ω	τ_Ω	e_m
$GL_2(\mathbb{Z}/p^N)$	$B_{1,1}$	$\mathbb{P}^1(\mathbb{Z}/p^N)$	$\{0,1,\dots,N\}$	$\tau^{(1)1}_{p(N)}$	$\varphi^{(1)1}_{p(N),m}$
$GL_{d+1}(\mathbb{Z}/p^N)$	$B_{d,1}$	$\mathbb{P}^d(\mathbb{Z}/p^N)$	$\{0,1,\dots,N\}$	$\tau^{(1)d}_{p(N)}$	$\varphi^{(1)d}_{p(N),m}$
$GL_{d_0+d_1}(\mathbb{Z}/p^N)$	$B_{d_0+d_1-1,1}$, B_{d_0,d_1}	–	$\{0,1,\dots,N\}$	$\tau^{(d_0)d_1}_{p(N)}$	$\varphi^{(d_0)d_1}_{p(N),m}$
$GL_2(\mathbb{Z}_p)$	$B_{1,1}$	$\mathbb{P}^1(\mathbb{Z}_p)$	$p^{\mathbb{N}} \cup \{0\}$	$\tau^{(1)1}_p$	$\varphi^{(1)1}_{p,m}$
$\mathrm{GL}_{d+1}(\mathbb{Z}_p)$	$B_{d,1}$	$\mathbb{P}^d(\mathbb{Z}_p)$	$p^{\mathbb{N}} \cup \{0\}$	$\tau^{(1)d}_p$	$\varphi^{(1)d}_{p,m}$
$GL_{d_0+d_1}(\mathbb{Z}_p)$	$B_{d_0+d_1-1,1}$, B_{d_0,d_1}	–	$p^{\mathbb{N}} \cup \{0\}$	$\tau^{(d_0)d_1}_p$	$\varphi^{(d_0)d_1}_{p,m}$
$GL_d(\mathbb{Z}_p)$	$B_{d-1,1}$, $B_{1,\dots,1}$	–	$\mathbb{N}^{r-1} \sqcup \cdots \sqcup \mathbb{N}^0$	$\tau^{(1,\dots,1)}_p$	$\varphi^{(1,\dots,1)}_{p,\overline{v}}$
$(\mathbb{Z}/p^N)^* \ltimes \mathbb{Z}/p^N$	$(\mathbb{Z}/p^N)^*$	$\mathbb{Z}/p^N$	$\{1,p,p^2,\dots,p^N=0\}$	$\tau^1_{p(N)}$	$\varphi^1_{p(N),m}$
$GL_d(\mathbb{Z}/p^N) \ltimes (\mathbb{Z}/p^N)^{\oplus d}$	$GL_d(\mathbb{Z}/p^N)$	$(\mathbb{Z}/p^N)^{\oplus d}$	$\{1,p,p^2,\dots,p^N=0\}$	$\tau^d_{p(N)}$	$\varphi^d_{p(N),m}$
$\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$	$\mathbb{Z}_p^*$	$\mathbb{Z}_p$	$p^{\mathbb{N}} \cup \{0\}$	τ^1_p	$\varphi^1_{p,m}$
$GL_d(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^{\oplus d}$	$GL_d(\mathbb{Z}_p)$	$\mathbb{Z}_p^{\oplus d}$	$p^{\mathbb{N}} \cup \{0\}$	τ^d_p	$\varphi^d_{p,m}$

Notice that the same results holds when we replace $\mathbb{Z}_p$ by $\mathcal{O}_K$ the ring of integers of a local field K, one only have to multiply the parameters α, β (or d_0, d_1) by r, where p^r is the number of elements in the residue field of $\mathcal{O}_K$. Similarly, the passage from the reals to the complex cases involves multiplying the parameters by $2 = [\mathbb{C};\mathbb{R}]$.

G	B	X	Ω	τ_Ω	e_m
O_2	$B_{1,1}$	$\mathbb{P}^1(\mathbb{R})$	$\{\pm 1\}\backslash\mathbb{P}^1(\mathbb{R}) = [0,\infty]$	$\tau^{1,1}_\eta$	$\varphi^{1,1}_{\eta,m}$
O_{d+1}	$B_{d,1}$	$\mathbb{P}^d(\mathbb{R})$	$[0,\infty]$	$\tau^{d,1}_\eta$	$\varphi^{d,1}_{\eta,m}$
$O_{d_0+d_1}$	$O_{d_0} \times O_{d_1}$, $O_{d_0+d_1-1} \times O_1$	–	$[0,\infty]$	τ^{d_0,d_1}_η	$\varphi^{d_0,d_1}_{\eta,m}$
O_d	$O_{d-1} \times O_1$, O_1^d	–	$\{\pm 1\}^d\backslash\mathbb{P}^{d-1}(\mathbb{R})$	$\tau^{(1,\dots,1)}_\eta$	$\varphi^{(1,\dots,1)}_{\eta,\overline{v}}$
U_2	$B_{1,1}$	$\mathbb{P}^1(\mathbb{C})$	$(\mathbb{C}^{(1)})\backslash\mathbb{P}^1(\mathbb{C}) = [0,\infty]$	$\tau^{2,2}_\eta$	$\varphi^{2,2}_{\eta,m}$
U_{d+1}	$B_{d,1}$	$\mathbb{P}^d(\mathbb{C})$	$[0,\infty]$	$\tau^{2d,2}_\eta$	$\varphi^{2d,2}_{\eta,m}$
$U_{d_0+d_1}$	$U_{d_0} \times U_{d_1}$, $U_{d_0+d_1-1} \times U_1$	–	$[0,\infty]$	$\tau^{2d_0,2d_1}_\eta$	$\varphi^{2d_0,2d_1}_{\eta,m}$
U_d	$U_{d-1} \times U_1$, U_1^d	–	$U_1^d\backslash\mathbb{P}^{d-1}(\mathbb{C})$	$\tau^{(2,\dots,2)}_\eta$	$\varphi^{(2,\dots,2)}_{\eta,\overline{v}}$

The proof of the p-adic case is just simple calculation of the relative measure $\tau^{(\alpha)\beta}_p$ showing that for the appropriate parameters α, β it is given by the Haar measure, more precisely, the image of the Haar measure on the corresponding Ω's. This is obtained by just counting. For example, Notice that $\varphi^{(\alpha)\beta}_{p,m}$ is orthogonal basis with respect to the measure $\tau^{(\alpha)\beta}_p$ and by our construction of the Markov chain $\varphi^{(\alpha)\beta}_{p,m}$ is defined mod p^m and not mod p^{m-1}.

The $\varphi_{p,m}^{(\alpha)\beta}$ are normalized by $\varphi_{p,m}^{(\alpha)\beta}(\mathrm{id}) = ||\varphi_{p,m}^{(\alpha)\beta}||^2_{H_p^{(\alpha)\beta}}$. Comparing it with the representation theory, we see that there exists a unique representation of $GL_2(\mathbb{Z}_p)$ in $L^2(X)$, which factors through $GL_2(\mathbb{Z}/p^m)$ but not through $GL_2(\mathbb{Z}/p^{m-1})$ (see [Hi]). All the finite dimensional representation of $GL_2(\mathbb{Z}_p)$ are representation of $GL_2(\mathbb{Z}/p^n)$ for some n.

For the real case, first of all we have to check the image of the Haar measure on the corresponding Ω coincide with $\tau_p^{\alpha,\beta}$. We notice that the basis $\varphi_{\eta,m}^{\alpha,\beta}$ is an eigenfunction of the Laplace–Beltrami operator Δ, which lies in the enveloping algebra and induces a G-invariant differential operator on $L^2(X)$. Identifying its radial part (i.e., the action of Δ on functions on Ω in the coordinate $x \in [0, \infty] \simeq [0, 1]$) with our operator D^+D, this gives the proof of real cases.

8

Real Grassmann Manifold

Summary. In Chap. 8 we give the classical theory of the decomposition of the representation afforded by the Grassmannian for the group $GL_d(\mathbb{Z}_\eta) = O_d$ (or U_d for the complex η). The space Ω_m^d parameterizes the relative positions of two m-planes. Let $\mathfrak{p}, \mathfrak{q} \subseteq \mathbb{R}^d$ be two planes and $m = \dim \mathfrak{p} \le \dim \mathfrak{q} \le \frac{1}{2}d$. Then we say that $\mathrm{typ}(\mathfrak{p}, \mathfrak{q}) = \{\theta_1, \ldots, \theta_m\}$, or the relative position (or the critical angles) of $\mathfrak{p}, \mathfrak{q}$ are $\{\theta_1, \ldots, \theta_m\}$ if and only if there exists orthonormal basis $A_1, \ldots, A_m$ for $\mathfrak{p}$ and $B_1, \ldots, B_n$ for $\mathfrak{q}$ such that $(A_i, B_j) = \delta_{ij} \cos \theta_i$. In other words, A_i and B_i have an angle θ_i and A_i is orthogonal to all B_j for $i \neq j$. Equivalently, if we consider the orthogonal projection from $\mathfrak{q}$ to $\mathfrak{p}$, $\mathrm{Pr}_{\mathfrak{p},\mathfrak{q}}$, and its adjoint $\mathrm{Pr}_{\mathfrak{q},\mathfrak{p}}$, since their composition $\mathrm{Pr}_{\mathfrak{p},\mathfrak{q}} \circ \mathrm{Pr}_{\mathfrak{q},\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ is self-adjoint, we have a diagonalization as

$$\mathrm{Pr}_{\mathfrak{p},\mathfrak{q}} \circ \mathrm{Pr}_{\mathfrak{q},\mathfrak{p}} \sim \begin{pmatrix} \cos^2 \theta_1 & & 0 \\ & \ddots & \\ 0 & & \cos^2 \theta_m \end{pmatrix}.$$

Notice that these two projections do not commute in general; if they do commute their composition is the orthogonal projection on the intersection $\mathfrak{p} \cap \mathfrak{q}$ (and $\cos^2 \theta_i \in \{0, 1\}$). We have calculated the Selberg measure on the space Ω_m^d of the relative positions. Then we determined the idempotents which are the multivariable (higher-rank) Jacobi polynomials.

8.1 Measures on the Higher Rank Spaces

8.1.1 Grassmann Manifolds

We start with the theory constructed by A. James and A. Constantine ([JC]). Recall that

η		real dimension
real	$O_d := \left\{ A \in \mathrm{Mat}_{d \times d}(\mathbb{R}) \,\middle|\, A^t \cdot A = I_d \right\}$	$\frac{1}{2}d(d-1)$
complex	$U_d := \left\{ A \in \mathrm{Mat}_{d \times d}(\mathbb{C}) \,\middle|\, A^* \cdot A = I_d \right\}$	$d(d-1)$

Note that the real dimension can be calculated by the numbers of equations (for the real case, we have $d^2 - \frac{1}{2}d(d+1) = \frac{1}{2}d(d-1)$). These are closed and compact subgroup of GL_d as in the p-adic case (notice that $GL_d(\mathbb{Z}_p)$ is the maximal compact subgroup of $GL_d(\mathbb{Q}_p)$ but is also open). We are interested in the Grassmann manifold;

$$X_m^d := \big\{m\text{-dimensional subspace of } \mathbb{R}^d \text{ (or } \mathbb{C}^d)\big\}.$$

It is easy to see that

η	Grassmann manifold	real dimension
real	$X_m^d(\mathbb{R}) = GL_d(\mathbb{R})/B_{m,d-m}(\mathbb{R}) = O_d/O_m \times O_{d-m}$	$m(d-m)$
complex	$X_m^d(\mathbb{C}) = GL_d(\mathbb{C})/B_{m,d-m}(\mathbb{C}) = U_d/U_m \times U_{d-m}$	$2m(d-m)$

For the real case, the Haar measure on X_m^d is the unique O_d-invariant probability measure and, hence, we get the unitary representation $O_d \to U(L^2(X_m^d))$. Similarly for the complex case.

Let us look at the quotient $O_d \backslash X_m^d \times X_m^d$. More generally if we want the intertwining operators between X_m^d and X_n^d, we consider $O_d \backslash X_m^d \times X_n^d$. Now we always assume $m \leq n \leq \frac{d}{2}$. Then it can be written as

$$O_d \backslash X_m^d \times X_n^d = O_m \times O_{d-m} \backslash O_d / O_n \times O_{d-n} \simeq \Omega_m.$$

As a set, we see that

$$\Omega_m = [0,1]^m/\mathfrak{S}_m = \big\{(y_1,\dots,y_m) \in [0,1]^m \,\big|\, 0 \leq y_1 \leq \cdots \leq y_m \leq 1\big\},$$

where $\mathfrak{S}^m$ is the symmetric group.

Take two planes $\mathfrak{p}_m \in X_m^d$ and $\mathfrak{p}_n \in X_n^d$. We look at $O_d(\mathfrak{p}_m, \mathfrak{p}_n)$ as the relative position of m-plane and n-plane. We denote by $\operatorname{typ}(\mathfrak{p}_m, \mathfrak{p}_n) = (y_1,\dots,y_m) \in \Omega_m$ if and only if there exists an orthonormal basis $X_1,\dots,X_m$ for $\mathfrak{p}_m$ and $\widetilde{X}_1,\dots,\widetilde{X}_n$ for $\mathfrak{p}_n$ such that $|(X_i, \widetilde{X}_j)| = \delta_{ij} y_i$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. In this case we say that $\mathfrak{p}_m$ and $\mathfrak{p}_n$ have the *critical angle* $y_i = |\cos\theta_{X_i,\widetilde{X}_i}|$. In other words, let us denote by $P_{\mathfrak{p}_n,\mathfrak{p}_m}$ the orthogonal projection $\mathrm{Pr}_{\mathfrak{p}_n|\mathfrak{p}_m} : \mathfrak{p}_n \to \mathfrak{p}_m$. Then the adjoint of $P_{\mathfrak{p}_n,\mathfrak{p}_m}$ is just given by $(P_{\mathfrak{p}_n,\mathfrak{p}_m})^* = P_{\mathfrak{p}_m,\mathfrak{p}_n}$. Note that the operator $P_{\mathfrak{p}_m,\mathfrak{p}_n} \circ P_{\mathfrak{p}_n,\mathfrak{p}_m} : \mathfrak{p}_n \to \mathfrak{p}_n$ is a positive definite self-adjoint operator, whence we can diagonalize it. Then eigenvalues of $P_{\mathfrak{p}_m,\mathfrak{p}_n} \circ P_{\mathfrak{p}_n,\mathfrak{p}_m}$ are just $\{y_1^2,\dots,y_m^2\}$.

In the case $n = m$, for $\mathfrak{p}, \mathfrak{p}' \in X_m^d$, it is easy to see that $\operatorname{typ}(\mathfrak{p},\mathfrak{p}') = \operatorname{typ}(\mathfrak{p}',\mathfrak{p})$. This gives the commutativity of the Hecke algebra $\mathcal{H}_m^d := \mathcal{S}(\Omega_m)$. Therefore we have a multiplicity free representation. We have the decomposition of $L^2(X_m^d)$ with respect to O_d. Similarly for U_d in the complex case.

8.1.2 Measures on O_m, X_m^d and V_m^d

Take an m-plane $\mathfrak{p} \in X_m^d$. Fix an orthonormal basis $\{a_1, \ldots, a_m\}$ for $\mathfrak{p}$. Take the vectors $\{b_1, \ldots, b_{d-m}\}$ with $b_j \in \mathbb{R}^d$ such that $\{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_{d-m}\}$ becomes orthonormal basis for $\mathbb{R}^d$. Then it is clear that $\{b_1, \ldots, b_{d-m}\}$ is the orthonormal basis for $\mathfrak{p}^\perp$ and we get the matrix $(A|B) = (a_1|\cdots|a_m|b_1|\cdots|b_{d-m}) \in O_d$. Now look at the following differential form

$$\overline{\tau}_m^d(\mathfrak{p}) := \prod_{\substack{1\le i\le m\\ 1\le j\le d-m}} \left|b_j^* \cdot da_i\right|.$$

This is a differential form of degree $m(d-m)$. Here $b_j^* da_i$ denotes the inner product; $b_j^* da_i = (da_i, b_j)$. It is independent on the choice of $\{a_i\}$ and $\{b_j\}$, and $\overline{\tau}_m^d$ is the O_d-invariant measure on X_m^d since the differential form of the highest degree determines the measure. Similarly $-\mathrm{Tr}(A^* dB \cdot B^* dA)$ (again this 2-form is independent on A and B by the property of the trace) is the O_d-invariant metric on X_m^d.

We next look at the space

$$\begin{aligned} V_m^d &= \big\{(a_1, \ldots, a_m) \,\big|\, \text{orthonormal basis for an } m\text{-dimensional} \\ &\qquad\quad \text{subspace of } \mathbb{R}^d \text{ (or } \mathbb{C}^d)\big\} \\ &= \big\{A \in \mathrm{Mat}_{d\times m}(\mathbb{R}) \text{ (or } \mathrm{Mat}_{d\times m}(\mathbb{C})) \,\big|\, A^* \cdot A = I_d\big\}. \end{aligned}$$

It can be also expressed as

η		real dimension
real	$V_m^d(\mathbb{R}) = O_d/O_{d-m}$	$m(d-m) + \frac{1}{2}m(m-1)$
complex	$V_m^d(\mathbb{C}) = U_d/U_{d-m}$	$2m(d-m) + m(m-1)$

Hence O_d acts on V_m^d and all stabilizers are isomorphic to O_{d-m}. We call V_m^d the space of m-frames in d-space. Let us take $A = (a_1, \ldots, a_m) \in V_m^d$ and $B = (b_1, \ldots, b_{d-m})$ such that $(A|B) \in O_d$. Similarly to the above, consider the following form

$$\tau_m^d(A) := \prod_{\substack{1\le i\le m\\ 1\le j\le d-m}} \left|b_j^* \cdot da_i\right| \cdot \prod_{1\le i<j\le m} \left|a_j^* \cdot da_i\right|.$$

This is independent of the choice of B and gives the $O_d \times O_m$-invariant measure, that is, $\tau_m^d(g_d A g_m) = \tau_m^d(A)$ for any $g_d \in O_d$ and $g_m \in O_m$.

For example, consider the case $m = 1$. Then V_1^d is just the $(d-1)$-sphere in $\mathbb{R}^d$ and

$$\tau_1^d(a) = \prod_{1\le j\le d-1} \left|b_j^* \cdot da\right|.$$

Fix a vector $e_d \in V_1^d$. Let us write $\mathbb{R}^d = \mathbb{R}^{d-1} \oplus \mathbb{R}e_d$ and $\mathrm{Pr}_{\mathbb{R}^{d-1}} : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the orthogonal projection with respect to the decomposition above. For

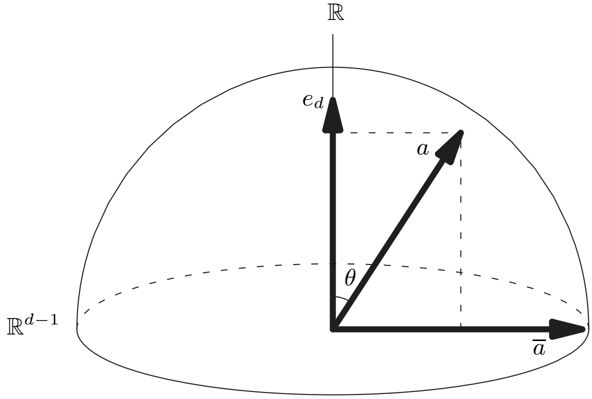


Fig. 8.1. A vector a

any vector $a \in \mathbb{R}^d$, we denote by $\overline{a}$ the image of the projection of a onto $\mathbb{R}^{d-1}$ normalized to be a unit vector;

$$\overline{a} := \frac{\mathrm{Pr}_{\mathbb{R}^{d-1}}(a)}{\left|\mathrm{Pr}_{\mathbb{R}^{d-1}}(a)\right|}.$$

Then it can be written as

$$a = \sin\theta \cdot \overline{a} + \cos\theta \cdot e_d,$$

where $\cos\theta = (a, e_d)$ (see Fig. 8.1).
It is easy to calculate that

$$da = (-\sin\theta \cdot e_d + \cos\theta \cdot \overline{a}) \cdot d\theta + \sin\theta \cdot d\overline{a} + \cos\theta \cdot de_d.$$

Now choose the vector b_1 as

$$b_1 := -\sin\theta \cdot e_d + \cos\theta \cdot \overline{a}$$

so that a and b_1 are orthogonal. Then we have

$$b_1^* da = (\sin^2\theta + \cos^2\theta) \cdot d\theta = d\theta.$$

Similarly, for $j > 1$, choose b_j as $b_j^* e_d = 0 = b_j^* \overline{a}$ so that $b_j^* \cdot da = \sin\theta b_j^* \cdot d\overline{a}$. We can obtain the explicit forms of the differential form by taking the product of all $b_j^* \cdot da$ as follows

$$\begin{aligned}\tau_1^d(a) = \prod_{1\le j\le d-1} \left|b_j^* \cdot da\right| &= |\sin\theta|^{d-2} \cdot d\theta \prod_{2\le j\le d-1} b_j^* \cdot d\overline{a} \\ &= |\sin\theta|^{d-2} \cdot d\theta \cdot \tau_1^{d-1}(\overline{a})\end{aligned}$$

$$= \cdots$$
$$= |\sin\theta_1|^{d-2}|\sin\theta_2|^{d-3}\cdots|\sin\theta_{d-2}|d\theta_1\cdots d\theta_{d-1}.$$

Here $\{\theta_j\}_{1\leq j\leq d-2}$ is the sequence of angles with $0<\theta_i<\pi$ for $1\leq j\leq d-2$ and $0<\theta_{d-1}<2\pi$. One can then calculate the following integral

$$\int_{V_1^d}\tau_1^d(a)=\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}=\frac{2}{\zeta_\eta(d)}.$$

Similarly, for the general m, the measure of V_m^d are obtained as follows; Choose an element $a_1\in V_1^d$ and let $a_1^\perp$ denote the $(d-1)$ dimensional subspace orthogonal to a_1. Then the measure τ_m^d on V_m^d is given by the product of two measures on V_1^d and V_{m-1}^{d-1}. Namely, we have

$$\tau_m^d(a)=\tau_1^d(a_1)\otimes\tau_{m-1}^{d-1}(a_1^\perp).$$

We also obtain the following integral by induction

$$\int_{V_m^d}\tau_m^d(a)=\int_{V_1^d}\tau_1^d(a_1)\int_{V_{m-1}^{d-1}}\tau_{m-1}^{d-1}(a)=\prod_{d-m<j\leq d}\frac{2}{\zeta_\eta(j)}.$$

Notice that the orthogonal group O_d is nothing but $O_d\equiv V_d^d$, that is, the case $m=d$. Hence, writing $\tau_d:=\tau_d^d$, we obtain

$$\int_{O_d}\tau_d=\prod_{1\leq j\leq d}\frac{2}{\zeta_\eta(j)}.$$

Let us calculate one more integral. We have the map $V_m^d\to X_m^d$ such that

$$(a_1,\ldots,a_m)\mapsto \mathrm{Span}_{\mathbb{R}}\{a_1,\ldots,a_m\}.$$

The group O_m acts on the space V_m^d, and $O_m\to V_m^d\to X_m^d$ is a fibration. Therefore the measure $\overline{\tau}_m^d$ on the Grassmann space X_m^d is obtained by $\tau_m^d=\tau_m\otimes\overline{\tau}_m^d$, whence

$$\int_{X_m^d}\overline{\tau}_m^d=\frac{\displaystyle\int_{V_m^d}\tau_m^d}{\displaystyle\int_{\mathrm{O}_m}\tau_m}=\frac{\displaystyle\prod_{1\leq j<m}\zeta_\eta(j)}{\displaystyle\prod_{d-m<j\leq d}\zeta_\eta(j)}.$$

Finally we normalize all the Haar measures τ_m, τ_m^d and $\overline{\tau}_m^d$ on O_m, V_m^d and X_m^d by dividing by the above values to get probability measures.

8.1.3 Measures on Ω_m

Let $\overline{\tau}^d_{m,n}$ be the probability measure on

$$\Omega_m = O_d \backslash X^d_m \times X^d_n = O_{m,d-m} \backslash O_d / O_{n,d-n}.$$

From this equation, $\overline{\tau}^d_{m,n}$ coincides with the image of the Haar measure $\overline{\tau}^d_m \otimes \overline{\tau}^d_n$ on $X^d_m \times X^d_n$ or the image of τ_d on O_d. Let us next calculate this measure by looking on the space $\mathrm{Mat}_{d\times m}(\mathbb{R})$. Put on $\mathrm{Mat}_{d\times m}(\mathbb{R})$ the normal low defined as

$$\phi(X) = e^{-\pi \mathrm{Tr}(X^* \cdot X)} dX, \qquad dX = \bigotimes_{\substack{1\le i\le d \\ 1\le j\le m}} dX_{ij},$$

where dX_{ij} is the additive Haar measure. This measure is an O_d-invariant probability measure on $\mathrm{Mat}_{d\times m}(\mathbb{R})$. Note that the set of all matrices $X \in \mathrm{Mat}_{d\times m}(\mathbb{R})$ satisfying $\mathrm{rank}(X) = m$ is of full measure with respect to the measure $\phi(X)$. On this set of full measure, we get the projection

$$\mathrm{pr} : \mathrm{Mat}_{d\times m}(\mathbb{R}) \to X^d_m$$

given by

$$X = (X_1, \dots, X_m) \longmapsto \mathrm{Span}_{\mathbb{R}}\{X_1, \dots, X_m\} \in X^d_m.$$

Then the projection $\mathrm{pr}_* \phi(X)$ of $\phi(X)$ onto the Grassmann manifold X^d_m is again the O_d-invariant probability measure and hence must coincide with the Haar measure $\overline{\tau}^d_m$ because of the uniqueness. Therefore we have $\overline{\tau}^d_m = \mathrm{pr}_* \phi(X)$.

Similarly, consider the space $\mathrm{Mat}_{d\times(m+n)}(\mathbb{R})$. Remark that $m + n \le d$ by assumption. We have also the probability measure $\phi(X|Y)$ where $X \in \mathrm{Mat}_{d\times m}(\mathbb{R})$ and $Y \in \mathrm{Mat}_{d\times n}(\mathbb{R})$. From the set of $\mathrm{rank}(X|Y) = m + n$ in $\mathrm{Mat}_{d\times(m+n)}(\mathbb{R})$, which is also of full measure with respect to $\phi(X|Y)$, we have the projection to $X^d_m \times X^d_n$. And we can take the type to Ω_m. Hence we have the map

$$t : \mathrm{Mat}_{d\times(m+n)}(\mathbb{R}) \xrightarrow{\mathrm{pr}} X^d_m \times X^d_n \xrightarrow{\mathrm{typ}} \Omega_m.$$

Since $\phi(X|Y) = \phi(X) \otimes \phi(Y)$, we have $\overline{\tau}^d_{m,n} = t_*(\phi(X) \otimes \phi(Y))$. Also we have

$$t_*\big(\phi(X) \otimes \phi(Y)\big) \equiv t_*(\phi(X) \otimes \delta_{Y_0}) \equiv t_*(\delta_{X_0} \otimes \phi(Y))$$

for any $Y_0 \in \mathrm{Mat}_{d\times m}(\mathbb{R})$ and any $X_0 \in \mathrm{Mat}_{d\times n}(\mathbb{R})$.

8.2 Explicit Calculations

8.2.1 Measures

Consider the following space

$$\Omega_m^* := \{(y_1, \dots, y_m) \in \Omega_m \,\big|\, 0 < y_1 < \cdots < y_m < 1\}.$$

This is a subset of Ω_m which is open, dense and of full measure because the set of y such that $y_1 = y_2$ or $y_1 = 0$ is measure 0. Therefore we need only to calculate the measure $\overline{\tau}^d_{m,n}$ on the space Ω^*_m. Fix $Y_0 \in \mathrm{Mat}_{d\times n}(\mathbb{R})$ and set $\mathfrak{q}_0 = \mathrm{pr}(Y_0) \equiv \mathbb{R}^n \in X^d_n$. Take $\mathfrak{p} \in X^d_m$ which is generic with respect to Y_0 in the sense $\mathrm{typ}(\mathfrak{p}, \mathfrak{q}_0) \in \Omega^*_m$. Hence we know that we have distinct eigenvalues for the composition of our projections. Now let $(a_1, \dots, a_m)$ be an orthonormal basis for $\mathfrak{p}$ with critical angle with $\mathfrak{q}_0$. Let $(\alpha_1, \dots, \alpha_m) \in V^n_m(\mathfrak{q}_0)$. Here we denote by $V^n_m(\mathfrak{q}_0)$ the set of all orthonormal sets in $\mathfrak{q}_0$. Choose the α_i's to have the critical angles. Namely, $\alpha_i \in \mathrm{pr}_{\mathfrak{q}_0} a_i$ with

$$\alpha_i^* \alpha_j = \delta_{ij}, \qquad (a_i, \alpha_j) = \delta_{ij} \cos\theta_i. \tag{8.1}$$

for some θ_i. Similarly, choose $(\beta_1, \dots, \beta_m) \in V^{d-n}_m(\mathfrak{q}_0^\perp)$ to have critical angle with $\mathfrak{p}$. Again we have $\beta_j \in \mathrm{pr}_{\mathfrak{q}_0^\perp} a_i$ with

$$\beta_i^* \beta_j = \delta_{ij}, \qquad (a_i, \beta_j) = \delta_{ij} \sin\theta_i. \tag{8.2}$$

Note that

$$\alpha_i^* \beta_j = 0 \tag{8.3}$$

Then we can write

$$a_i = \cos\theta_i \cdot \alpha_i + \sin\theta_i \cdot \beta_i. \tag{8.4}$$

Differentiating these equations above, we have

$$da_i = \big(-\sin\theta_i \alpha_i + \cos\theta_i \beta_i\big) \cdot d\theta_i + \cos\theta_i \cdot d\alpha_i + \sin\theta_i \cdot d\beta_i, \tag{8.5}$$

and

$$\begin{cases} \alpha_i^* \cdot d\alpha_j + \alpha_j^* \cdot d\alpha_i = 0, \\ \beta_i^* \cdot d\beta_j + \beta_j^* \cdot d\beta_i = 0 \end{cases} (i \neq j), \quad \begin{cases} \alpha_i^* \cdot d\alpha_i = 0, \\ \beta_i^* \cdot d\beta_i = 0, \end{cases} \quad \begin{cases} \alpha_i^* \cdot d\beta_j = 0, \\ \beta_i^* \cdot d\alpha_j = 0. \end{cases} \tag{8.6}$$

Now, choose the basis $b_1, \dots, b_{d-m}$ for $\mathfrak{p}^\perp$ as follows:

$$b_j = -\sin\theta_j \alpha_j + \cos\theta_j \beta_j \quad (1 \le j \le m). \tag{8.7}$$

Note that $\mathrm{Pr}_{\mathfrak{q}_0}\{b_1, \dots, b_m\} = \{\alpha_1, \dots, \alpha_m\} = \mathrm{Pr}_{\mathfrak{q}_0}(\mathfrak{p})$. Choose $b_j \in \mathfrak{q}_0$ for $j = m+1, \dots, n$ such that

$$b_j^* \alpha_i = 0 \qquad (1 \le i \le m). \tag{8.8}$$

Similarly, $\mathrm{Pr}_{\mathfrak{q}_0^\perp}\{b_1, \dots, b_{d-m}\} = \{\beta_1, \dots, \beta_m\} = \mathrm{Pr}_{\mathfrak{q}_0^\perp}(\mathfrak{p})$ and we can choose $b_j \in \mathfrak{q}_0^\perp$ for $j = n+1, \dots, d-m$ such that

$$b_j^* \beta_i = 0 \qquad (1 \le i \le m). \tag{8.9}$$

We get

$$\begin{cases} b_i^* da_i = d\theta_i & 1 \le i \le m, \\ b_i^* da_j = -\sin\theta_i \cos\theta_j \cdot \alpha_i^* d\alpha_j + \cos\theta_i \sin\theta_j \cdot \beta_i^* d\beta_j & 1 \le i, j \le m, i \ne j. \end{cases} \tag{8.10}$$

Take the product of two such, we have

$$b_i^* da_j \wedge b_j^* da_i = \left(\cos^2\theta_j - \cos^2\theta_i\right) \cdot \alpha_j^* d\alpha_i \wedge \beta_j^* d\beta_i \qquad (1 \le i, j \le m,\ i \ne j). \tag{8.11}$$

In fact, it is easy to see that

$$\begin{aligned} b_i^* da_j \wedge b_j^* da_i &= \left(-\sin\theta_i \cos\theta_j \cdot \alpha_i^* d\alpha_j + \cos\theta_i \sin\theta_j \cdot \beta_i^* d\beta_j\right) \\ &\quad \wedge \left(-\sin\theta_j \cos\theta_i \cdot \alpha_j^* d\alpha_i + \cos\theta_j \sin\theta_i \cdot \beta_j^* d\beta_i\right) \\ &= -\sin^2\theta_i \cos^2\theta_j \cdot \alpha_i^* d\alpha_j \wedge \beta_j^* d\beta_i - \sin^2\theta_j \cos^2\theta_i \cdot \beta_i^* d\beta_j \wedge \alpha_j^* d\alpha_i \\ &= \left(\sin^2\theta_i \cos^2\theta_j - \sin^2\theta_j \cos^2\theta_i\right) \cdot \alpha_j^* d\alpha_i \wedge \beta_j^* d\beta_i \\ &= \left(\cos^2\theta_j - \cos^2\theta_i\right) \cdot \alpha_j^* d\alpha_i \wedge \beta_j^* d\beta_i. \end{aligned}$$

Similarly we have

$$\begin{cases} b_j^* da_i = b_j^* d\alpha_i \cdot \cos\theta_i & (j = m+1, \ldots, n, \quad 1 \le i \le m), \\ b_j^* da_i = b_j^* d\beta_i \cdot \sin\theta_i & (j = n+1, \ldots, d-m, \quad 1 \le i \le m), \end{cases} \tag{8.12}$$

Then the Haar measure $\overline{\tau}_m^d$ on the Grassmann manifold X_m^d can be calculated as

$$\begin{aligned} \overline{\tau}_m^d(\mathfrak{p}) &= \prod_{\substack{1 \le i \le m \\ 1 \le j \le d-m}} b_j^* da_i \\ &= \prod_{1 \le i \le m} b_i^* da_i \cdot \prod_{1 \le i < j \le m} b_i^* da_j \wedge b_j^* da_i \cdot \prod_{\substack{1 \le i \le m \\ m < j \le n}} b_j^* da_i \cdot \prod_{\substack{1 \le i \le m \\ n < j \le d-m}} b_j^* da_i \\ &= \prod_{1 \le i \le m} d\theta_i \prod_{1 \le i < j \le m} \Big(\left(\cos^2\theta_j - \cos^2\theta_i\right) \cdot \alpha_j^* d\alpha_i \wedge \beta_j^* d\beta_i\Big) \\ &\quad \otimes \prod_{\substack{1 \le i \le m \\ m < j < n}} b_j^* d\alpha_i \cdot \cos\theta_i \prod_{\substack{1 \le i \le m \\ n < j \le d-m}} b_j^* d\beta_i \cdot \sin\theta_i. \end{aligned}$$

Hence we have

$$\begin{aligned} \overline{\tau}_m^d(\mathfrak{p}) &= \Big[\prod_{i < j \le m} \alpha_j^* d\alpha_i \cdot \prod_{\substack{i \le m \\ m < j \le n}} b_j^* d\alpha_i\Big] \otimes \Big[\prod_{i < j \le m} \beta_j^* d\beta_i \cdot \prod_{n < j \le d-m} b_j^* d\beta_i\Big] \\ &\quad \otimes \Big[\prod_{i \le m} (\cos\theta_i)^{n-m} (\sin\theta_i)^{d-n-m} \cdot \prod_{1 \le i < j \le m} (\cos^2\theta_j - \cos^2\theta_i) \cdot d\theta_1 \cdots d\theta_m\Big]. \end{aligned} \tag{8.13}$$

We have the decomposition of the Grassmann manifolds X_m^d (modulo measure 0):

$$X_m^d \simeq V_m^n(\mathfrak{q}_0) \times V_m^{d-n}(\mathfrak{q}_0^{\perp}) \times \Omega_m.$$

Then the measure $\overline{\tau}_m^d$ on X_m^d is given by the product of three measures with respect to the decomposition above. Namely, we have

$$\overline{\tau}_m^d = \tau_m^n \otimes \tau_m^{d-n} \otimes \overline{\tau}_{m,n}^d,$$

where τ_m^n, τ_m^{d-n} and $\overline{\tau}_{m,n}^d$ are the measure on $V_m^n(\mathfrak{q}_0)$, $V_m^{d-n}(\mathfrak{q}_0^\perp)$ and Ω_m respectively given by

$$\tau_m^n = \prod_{i<j\leq m} \alpha_j^* d\alpha_i \cdot \prod_{\substack{i\leq m \\ m<j\leq n}} b_j^* d\alpha_i,$$

$$\tau_m^{d-n} = \prod_{i<j\leq m} \beta_j^* d\beta_i \cdot \prod_{\substack{1<i\leq m \\ n<j\leq d-m}} b_j^* d\beta_i,$$

$$\overline{\tau}_{m,n}^d = \prod_{i\leq m} (\cos\theta_i)^{n-m} (\sin\theta_i)^{d-n-m} \cdot \prod_{1\leq i<j\leq m} (\cos^2\theta_j - \cos^2\theta_i) \cdot d\theta_1 \cdots d\theta_m.$$

Note that $\overline{\tau}_{m,n}^d$ is unnormalized. Let us normalize $\overline{\tau}_{m,n}^d$ to be a probability measure as follows;

$$\overline{\tau}_{m,n}^d := \prod_{1\leq j\leq m} \frac{2\zeta_\eta(d-m+j)}{\zeta_\eta(j)\zeta_\eta(n-m+j)\zeta_\eta(d-n-m+j)}$$
$$\times \prod_{i\leq m} (\cos\theta_i)^{n-m} (\sin\theta_i)^{d-n-m} \cdot \prod_{i<j\leq m} (\cos^2\theta_j - \cos^2\theta_i) \cdot d\theta_1 \cdots d\theta_m.$$

Changing to the variable $y_i = \cos^2\theta_i$. Then, since $dy_i = 2\cos\theta_i \sin\theta_i \cdot d\theta_i$, we have

$$= \prod_{1\leq j\leq m} \frac{\zeta_\eta(d-m+j)}{\zeta_\eta(j)\zeta_\eta(n-m+j)\zeta_\eta(d-n-m+j)}$$
$$\times \prod_{i\leq m} y_i^{\frac{n-m}{2}-1} (1-y_i)^{\frac{d-n-m}{2}-1} \prod_{i<j\leq m} |y_j - y_i| dy_1 \cdots dy_m.$$

This is the special case of the Selberg measure (see the next section). Consequently we have

$$\overline{\tau}_{m,n}^d = \prod_{1\leq j\leq m} \frac{\zeta_\eta(d-m+j)}{\zeta_\eta(j)\zeta_\eta(n-m+j)\zeta_\eta(d-n-m+j)}$$
$$\times |\det Y|^{\frac{n-m}{2}-1} |\det(I-Y)|^{\frac{d-n-m}{2}-1} dY. \tag{8.14}$$

Here Y is the orthogonal projection from $\mathfrak{p}$ to $\mathfrak{q}_0$ followed by the orthogonal projection from $\mathfrak{q}_0$ to $\mathfrak{p}$. We call $\overline{\tau}_{m,n}^d$ the higher rank β-measure.

8.2.2 Metrics

We saw that $-\mathrm{Tr}(A^* dB \cdot B^* dA)$ is the O_d-invariant metric. Let again $\mathfrak{p} = \mathrm{Span}_\mathbb{R} A \in X_m^d$ and $\mathfrak{p}^\perp = \mathrm{Span}_\mathbb{R} B$ which is generic with respect to a fixed

$\mathfrak{q}_0 \in X_n^d$. Then we can write A as some orthogonal matrix of size $d \times m$ and B of size $d \times (d-m)$ as

$$A = \left(\begin{array}{c|c} \mathfrak{u}_n & 0 \\ \hline 0 & \mathfrak{u}_{d-n} \end{array}\right) \left(\begin{array}{c} \cos\theta I_m \\ 0_{n-m,m} \\ \hline \sin\theta I_m \\ 0_{d-n-m,m} \end{array}\right), \quad B = \left(\begin{array}{c|c} \mathfrak{u}_n & 0 \\ \hline 0 & \mathfrak{u}_{d-n} \end{array}\right) \left(\begin{array}{cc|c} -\sin\theta I_m & 0 & 0 \\ 0 & I_{n-m} & 0 \\ \hline \cos\theta I_m & 0 & 0 \\ 0 & 0 & I_{d-n-m} \end{array}\right)$$

for some $\mathfrak{u}_n \in O_n$ and $\mathfrak{u}_{d-n} \in O_{d-n}$ where

$$\cos\theta I_m := \begin{pmatrix} \cos\theta_1 & & 0 \\ & \ddots & \\ 0 & & \cos\theta_m \end{pmatrix}, \quad \sin\theta I_m := \begin{pmatrix} \sin\theta_1 & & 0 \\ & \ddots & \\ 0 & & \sin\theta_m \end{pmatrix}.$$

This means that the O_d-invariant metric $-\mathrm{Tr}(A^* dB \cdot B^* dA)$ can be expressed as

$$-\mathrm{Tr}(A^* dB \cdot B^* dA) = (d\theta_1)^2 + \cdots + (d\theta_m)^2 + (\text{terms in } d\mathfrak{u}_n \text{ and } d\mathfrak{u}_{d-n})$$

Here, we are not interested in this lower terms because we want the radial part of the Laplacian. Similarly, changing to the variable $y_i = \cos^2\theta_i$, we have

$$\begin{aligned} &= \sum_{1\le i\le m} \frac{dy_i^2}{4y_i(1-y_i)} + (\text{terms in } d\mathfrak{u}_n \text{ and } d\mathfrak{u}_{d-n}) \\ &=: \sum_{1\le i,j\le m} g_{ij}\, dy_i dy_j. \end{aligned}$$

We also know the determinant of the metric $g = (g_{ij})$, which is given by the measure;

$$(\det g)^{\frac{1}{2}} = \prod_i y_i^{\frac{n-m}{2}-1}(1-y_i)^{\frac{d-n-m}{2}-1} \prod_{1\le i<j\le m} |y_j - y_i|$$

and the Laplace–Beltrami operator is given by the formula

$$\Delta = -\frac{1}{4}(\det g)^{\frac{1}{2}} \sum_{1\le i\le m} \frac{\partial}{\partial y_i}(\det g)^{-\frac{1}{2}} g^{-1} \frac{\partial}{\partial y_i} = D - \delta,$$

where,

$$D = D^* + \big(\frac{d}{2} - m + 1\big)E,\ D^* = \sum_{1\le i\le m} {y_i}^2 \frac{\partial^2}{\partial {y_i}^2} + \sum_{i\ne j} {y_i}^2 \frac{1}{y_j - y_i}\frac{\partial}{\partial y_i},$$

$$E = \sum_{1\le i\le m} y_i \frac{\partial}{\partial y_i},$$

$$\delta = \delta^* + \big(\frac{q}{2} - m + 1\big)\mathcal{E}, \qquad \delta^* = \sum_{1\le i\le m} y_i \frac{\partial^2}{\partial {y_i}^2} + \sum_{i\ne j} y_i \frac{1}{y_j - y_i}\frac{\partial}{\partial y_i},$$

$$\mathcal{E} = \sum_{1\le i\le m} \frac{\partial}{\partial y_i}.$$

Note that D^* and δ^* are independent of the parameters. Acting on polynomials in y_i, D^* and E preserve the degree, but δ^* and $\mathcal{E}$ reduce the degree by one. That is the important points.

8.3 Higher Rank Orthogonal Polynomials

8.3.1 Real Case

Let $\mathrm{Sym}^k(P_m)$ be the space of the homogenous polynomials of degree k in symmetric matrices of size $m \times m$. This is the finite dimensional representation of $GL_m(\mathbb{R})$ via the action

$$GL_m(\mathbb{R}) \ni g \longmapsto (Y \mapsto gYg^t).$$

It decomposed in terms of the highest weight representation

$$\mathrm{Sym}^k(P_m) \simeq \bigoplus_{\substack{\lambda \in \Lambda_m \\ |\lambda|=k, \lambda_1' \leq m}} V_\lambda,$$

where $\Lambda_m := \left\{(\lambda_1, \ldots, \lambda_m) \,\middle|\, \lambda_1 \geq \cdots \geq \lambda_m\right\}$. Let $y_1, \ldots, y_m$ be the eigenvalues of Y. Then the trace $\mathrm{Tr}(Y)^k$ is the symmetric function in $y_1, \ldots, y_m$ and can be written as

$$\mathrm{Tr}(Y)^k = \sum_\lambda C_\lambda(Y),$$

where $C_\lambda(Y)$ is also a symmetric polynomial in $y_1, \ldots, y_m$, which we call the zonal polynomial, and is a spherical function on $GL_m(\mathbb{R})/O_m$. Note that $GL_m(\mathbb{R})/O_m$ is not a compact space. Normalize $C_\lambda(Y)$ as

$$C_\lambda^*(Y) := \frac{C_\lambda(Y)}{C_\lambda(I)}.$$

Define the binomial coefficient $\binom{\lambda}{v}$ for partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\nu = (\nu_1, \ldots, \nu_m)$ by

$$C_\lambda^*(I + Y) = \sum_{\nu \leq \lambda} \binom{\lambda}{\nu} C_\nu^*(Y).$$

Here $\nu \leq \lambda$ means that $\nu_i \leq \lambda_i$ for all $1 \leq i \leq m$. If we put $Y := \frac{2}{d}Y$ in the higher rank β-measure and take the limit $d \to \infty$, then we get the probability measure τ_m^n on P_m^+ where P_m^+ is the set of all positive definite symmetric matrices of size $m \times m$;

$$\tau_m^n(Y) := \frac{1}{\zeta_m(n)} e^{-\mathrm{Tr}(Y)} |\det Y|^{\frac{n-m-1}{2}} dY,$$

where $\zeta_m(n)$ is the multi-variables zeta function defined by

$$\zeta_m(n) := \int_{P_m^+} e^{-\mathrm{Tr}Y} |\mathrm{det}Y|^{\frac{n-m-1}{2}} dY = \pi^{\frac{m(m-1)}{4}} \prod_{1 \le i \le m} \Gamma\Big(\frac{n-i+1}{2}\Big).$$

We call τ_m^n the higher rank Laguerre measure.

The higher rank Laguerre polynomial of parameter n is defined by

$$\varphi_\lambda^n(Y) := {}_1F_1(-\lambda; \frac{n}{2}; Y) = \sum_{\nu \le \lambda} (-1)^{|\nu|} \binom{\lambda}{\nu} \frac{1}{\left(\frac{n}{2}\right)_\nu} C_\nu^*(Y),$$

where $\lambda \in \Lambda_m$ and

$$\Big(\frac{n}{2}\Big)_\nu := \prod_{1 \le i \le m} \Big(\frac{n-i+1}{2}\Big)\Big(\frac{n-i+1}{2}+1\Big) \cdots \Big(\frac{n-i+1}{2}+\nu_i-1\Big).$$

Then φ_λ^n is orthogonal with respect to the limit measure τ_m^n. Namely,

$$\int_{P_m^+} \varphi_\lambda^n(Y) \varphi_{\lambda'}^n(Y) \tau_m^n(Y) = \delta_{\lambda,\lambda'} \cdot \frac{k!}{C_\lambda(I)\left(\frac{n}{2}\right)_\lambda}.$$

The highest term of the polynomial $\varphi_\lambda^n(Y)$ is given by

$$\varphi_\lambda^n(Y) = \frac{(-1)^{|\lambda|}}{\left(\frac{n}{2}\right)_\lambda} C_\lambda^*(Y) + (\text{terms of lower degrees}). \tag{8.15}$$

It follows from the asymptotic (8.15) that for the intertwining operators $\varphi_\lambda^{d,m,n}(Y) : X_m^d \to X_n^d$ (or idempotent if $n = m$) we have $\varphi_\lambda^{d,m,n} \to \varphi_\lambda^n$ as $d \to \infty$, where φ_λ^n is orthogonal with respect to the higher rank Laguerre measure. We call $\varphi_\lambda^{d,m,n}$ the higher rank multi-variable Jacobi polynomial.

Again it can be written as

$$\varphi_\lambda^{d,m,n} = (\text{const.}) \cdot C_\lambda^*(\mathrm{diag}(y_1, \dots, y_m)) + (\text{terms of lower degrees}).$$

The function $\varphi_\lambda^{d,m,n}$ is also an eigen-function of the Laplacian;

$$\Delta \varphi_\lambda^{d,m,n} = d_{\lambda,n}^d \cdot \varphi_\lambda^{d,m,n}.$$

Remember that the Laplacian has two part; $\Delta = D - \delta$ where D preserves degrees and δ reduces degrees. If we consider the action of these operators, we see that the homogeneous part of the highest degree must be an eigen-function of the operator D, with the same eigenvalues. Now, the actions is given as follows;

$$DC_\lambda^* = d_{\lambda,n}^d C_\lambda^*, \qquad D^* C_\lambda^* = \big(\rho_\lambda + |\lambda|(m-1)\big) C_\lambda^*, \qquad EC_\lambda^* = |\lambda| C_\lambda^*,$$

where $\rho_\lambda := \sum_{1\le i\le m} \lambda_i(\lambda_i - i)$. Hence we know that

$$\Delta\varphi_\lambda^{d,m,n} = \big(\rho_\lambda + |\lambda|\frac{d}{2}\big)\varphi_\lambda^{d,m,n}$$

since $D = D^* + \big(\frac{d}{2} - m + 1\big)E$. This shows that we have different eigenvalues for different λ.

For the operator $\mathcal{E}$ it is very easy to see how it acts on C_λ^*;

$$\begin{aligned}\mathcal{E}C_\lambda^*(\mathrm{diag}(y_1,\dots,y_m)) &= \sum_i \frac{\partial}{\partial y_i} C_\lambda^*(\mathrm{diag}(y_1,\dots,y_m))\\ &= \lim_{\varepsilon\to 0} \frac{C_\lambda^*(\varepsilon I_m + \mathrm{diag}(y_1,\dots,y_m)) - C_\lambda^*(\mathrm{diag}(y_1,\dots,y_m))}{\varepsilon}\\ &= \sum_{\substack{i\\ \lambda_i>\lambda_{i+1}}} \binom{\lambda}{\lambda(i-)} C_{\lambda(i-)}^*(\mathrm{diag}(y_1,\dots,y_m)),\end{aligned}$$

where $\lambda(i-) := (\lambda_1,\dots,\lambda_i - 1,\dots,\lambda_m)$. Also we know the action of δ^*, which is essentially the commutator in the sense $\delta^* = \frac{1}{2}(\mathcal{E}D^* - D^*\mathcal{E})$;

$$\delta^* C_\lambda^* = \sum_{1\le i\le m} \Big(\lambda_i + \frac{m-i}{2} - 1\Big)\binom{\lambda}{\lambda(i-)} C_{\lambda(i-)}^*.$$

Hence we see that the idempotent $\varphi_\lambda^{d,m,n}$ can be written as

$$\varphi_\lambda^{d,m,n} = \sum_{\nu\le\lambda} (-1)^{|\nu|}\binom{\lambda}{\nu}\frac{A^d_{\lambda,\nu(i+)}}{\left(\frac{n}{2}\right)_\nu} C_\nu^*$$

with some coefficient $A^d_{\lambda,n}$. The eigen function equation for $\varphi_\lambda^{d,m,n}$ by the action of the Laplacian translates into the coefficient $A^d_{\lambda,\nu}$ satisfying the following recursion equation;

$$A^d_{\lambda,\nu} = \sum_{\nu_{i-1}>\nu_i} \frac{\binom{\lambda}{\nu(i+)}\binom{\nu(i+)}{\nu}}{\binom{\lambda}{\nu}} \frac{A^d_{\lambda,\nu(i+)}}{\big((|\lambda| - |\nu|)\frac{d}{2} + \rho_\lambda - \rho_\nu\big)}.$$

This, in principal, determines the idempotents. Similarly, for the complex case, we just shift the parameters by 2.

8.3.2 General Case

More generally, we take the following measure $\tau_\eta^{\alpha,\beta,\gamma}$ with parameters $\alpha, \beta, \gamma > 0$ which is normalized to be a probability measure;

$$\tau_\eta^{\alpha,\beta,\gamma} := \big(C_\eta^{\alpha,\beta,\gamma}\big)^{-1} \prod_{1\le i\le m} y_i^{\frac{\beta}{2}-1}(1-y_i)^{\frac{\alpha}{2}-1} \prod_{i<j} |y_j - y_i|^\gamma dy,$$

This is called the Selberg measure. All of our higher rank β-measures are special cases of this measure for special parameters. Namely,

$$\overline{\tau}^d_{m,n} = \begin{cases} \tau_\eta^{d-n-m+1,n-m+1,1} & \text{if } \eta \text{ is real,} \\ \tau_\eta^{2(d-n-m+1),2(n-m+1),2} & \text{if } \eta \text{ is complex.} \end{cases}$$

The constant $C_\eta^{\alpha,\beta,\gamma}$ is the higher rank beta function, it was determined by Selberg ([Se]) and explicitly given by

$$\left(C_\eta^{\alpha,\beta,\gamma}\right)^{-1} := \prod_{j\leq m} \frac{\Gamma\left(\frac{\alpha+\beta}{2} + (m+j-2)\gamma\right)\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\beta}{2} + (j-1)\frac{\gamma}{2}\right)\Gamma\left(\frac{\alpha}{2} + (j-1)\frac{\gamma}{2}\right)\Gamma\left(j\cdot\frac{\gamma}{2}\right)}. \tag{8.16}$$

Let $\varphi_\lambda^{\alpha,\beta,\gamma}(y_1,\dots,y_m)$ be the orthogonal polynomials with respect to $\tau_\eta^{\alpha,\beta,\gamma}$ induced by the partition $\lambda = (\lambda_1,\dots,\lambda_m)$. We call it the multi-variable Jacobi polynomial. It is a symmetric polynomial in $y_1,\dots,y_m$ and satisfies

$$\varphi_\lambda^{\alpha,\beta,\gamma} = (\text{const.})\cdot \mathfrak{m}_\lambda + (\text{terms of lower degrees}),$$

where $\mathfrak{m}_\lambda = S_m(y_1^{\lambda_1},\dots,y_m^{\lambda_m})$ is the monomial symmetric function. These are orthogonal with respect to the measure $\tau_\eta^{\alpha,\beta,\gamma}$, that is,

$$(\varphi_\lambda^{\alpha,\beta,\gamma}, \mathfrak{m}_\nu)_{\tau_\eta^{\alpha,\beta,\gamma}} = 0 \quad \text{for } \nu < \lambda.$$

We need the following normalization since it is also determined up to a constant;

$$||\varphi_\lambda^{\alpha,\beta,\gamma}||^2 = \varphi_\lambda^{\alpha,\beta,\gamma}(0).$$

The function $\varphi_\lambda^{\alpha,\beta,\gamma}$ is also an eigen-function of some differential operator $\Delta^{\alpha,\beta,\gamma}$. Similarly if we put $y_i := \frac{2}{\alpha}y_i$ and take the limit $\alpha\to\infty$, we have $\varphi_\lambda^{\alpha,\beta,\gamma} \to \varphi_\lambda^{\infty,\beta,\gamma}$ where $\varphi_\lambda^{\infty,\beta,\gamma}$ is the multi-variable Laguerre polynomial. Similarly, $\varphi_\lambda^{\infty,\beta,\gamma}$ is orthogonal with respect to the measure

$$\tau_\eta^{\infty,\beta,\gamma} := \left(C_\eta^{\infty,\beta,\gamma}\right)^{-1} \prod_{1\leq i\leq m} y_i^{\frac{\beta}{2}-1} e^{-y_i} \prod_{i<j} |y_j - y_i|^\gamma dy.$$

9

p-Adic Grassmann Manifold

Summary. In Chap. 9 we give the analogous theory over the p-adic, giving the decomposition of the representation of $GL_d(\mathbb{Z}_p)$ afforded by the p-adic Grassmannian. The relative position of two planes $\mathfrak{p}, \mathfrak{q} \subseteq \mathbb{Z}_p^d$ is given by the type of the $\mathbb{Z}_p$-module $\mathfrak{p} \cap \mathfrak{q}$, i.e., by a partition. We calculate the measure on Ω_m^d, and describe the idempotents - the p-adic multivariable Jacobi polynomials.

9.1 Representation of $GL_d(\mathbb{Z}_p)$

9.1.1 Measures on $GL_d(\mathbb{Z}_p)$, V_m^d and X_m^d

Let p be a finite prime. First of all, we see that $GL_d(\mathbb{Z}_p)$ is expressed as the inverse limit;

$$GL_d(\mathbb{Z}_p) = \varprojlim G_{N^d},$$

where $G_{N^d} := GL_d(\mathbb{Z}/p^N)$. Then we obtain the following diagram by the determinant;

$$\begin{array}{ccc} \mathrm{Mat}_{d\times d}(\mathbb{Z}_p) & \xrightarrow{\det} & \mathbb{Z}_p \\ \cup & & \cup \\ GL_d(\mathbb{Z}_p) & \xrightarrow{\det} & \mathbb{Z}_p^* \end{array}$$

Note that $GL_d(\mathbb{Z}_p)$ is the maximal compact subgroup of $GL_d(\mathbb{Q}_p)$. This is similar to the real case. Namely, O_d is the maximal compact subgroup of $GL_d(\mathbb{R})$ and U_d is of $GL_d(\mathbb{C})$. But unlike the real case (where O_d and U_d are *closed* subset of $\mathrm{Mat}_{d\times d}$), the above diagram shows that $GL_d(\mathbb{Z}_p)$ is an *open* subset of $\mathrm{Mat}_{d\times d}(\mathbb{Z}_p)$. Now we have the measure on $\mathrm{Mat}_{d\times d}(\mathbb{Z}_p)$ defined by the additive Haar measure

$$dx := \bigotimes_{1\le i,j\le d} dx_{ij}.$$

This measure satisfies $d(gx) = |\det g|dx$ for $g \in \mathrm{Mat}_{d\times d}(\mathbb{Z}_p)$. In particular, dx is $GL_d(\mathbb{Z}_p)$-invariant measure on $\mathrm{Mat}_{d\times d}(\mathbb{Z}_p)$.

Let $A_1, \ldots, A_m \in \mathbb{Z}_p^{\oplus d} \subseteq \mathbb{Q}_p^{\oplus d}$. We call $A_1, \ldots, A_m$ orthonormal if

$$\mathbb{Z}_p^{\oplus d} \Big/ \sum_{1\le i\le m} \mathbb{Z}_p A_i \simeq \mathbb{Z}_p^{\oplus(d-m)}$$

as $\mathbb{Z}_p$-module. Let $\overline{A}_i$ be the image of A_i modulo p for $1 \le i \le m$. Then $A_1, \ldots, A_m$ are orthonormal if and only if $\overline{A}_1, \ldots, \overline{A}_m \in \mathbb{F}_p^{\oplus d}$ are linearly independent over $\mathbb{F}_p$. This is also equivalent to the existence of $B_1, \ldots, B_{d-m} \in \mathbb{Z}_p^{\oplus d}$ such that $(A_1, \ldots, A_m | B_1, \ldots, B_{d-m}) \in GL_d(\mathbb{Z}_p)$. Then we denote by

$$V_m^d := \left\{ A = (A_1, \ldots, A_m) \in \mathrm{Mat}_{d\times m}(\mathbb{Z}_p) \,\middle|\, A_1, \ldots, A_m \text{ are orthonormal.} \right\}.$$

The group $GL_d(\mathbb{Z}_p)$ acts on V_m^d transitively and the stabilizer of the standard basis $1 = (E_1, \ldots, E_m)$ is given by $GL_{d-m}(\mathbb{Z}_p) \ltimes \mathrm{Mat}_{m\times(d-m)}(\mathbb{Z}_p)$. Hence it holds that

$$V_m^d \simeq GL_d(\mathbb{Z}_p)/GL_{d-m}(\mathbb{Z}_p) \ltimes \mathrm{Mat}_{m\times(d-m)}(\mathbb{Z}_p).$$

Note that the factor $\mathrm{Mat}_{m\times(d-m)}(\mathbb{Z}_p)$ does not appear in the real case. Let us first consider the case of $m = 1$. It is easy to see that

$$V_1^d = \left\{ A \in \mathbb{Z}_p^{\oplus d} \,\middle|\, |A|_p = 1 \right\},$$

where $|A|_p = |{}^t(a_1, \ldots, a_d)|_p := \max_{1\le i\le d} |a_i|_p$. Then the condition $|A|_p = 1$ is equivalent to $\overline{A} \not\equiv 0$ modulo p. The measure of V_1^d can be calculated as follows;

$$\begin{aligned}
\int_{V_1^d} dx &= (1 - p^{-1}) + p^{-1} \int_{V_1^{d-1}} dx = \cdots \\
&= (1 - p^{-1}) + p^{-1}(1 - p^{-1}) + p^{-2}(1 - p^{-1}) + \cdots + p^{-(d-1)}(1 - p^{-1}) \\
&= 1 - p^{-d} \\
&= \frac{1}{\zeta_p(d)}.
\end{aligned}$$

Similarly, for general $m \ge 1$, we have

$$\int_{V_m^d} dx = \int_{V_1^d} dx \int_{V_{m-1}^{d-1}} dx = \cdots = \prod_{d-m<j\le d} \frac{1}{\zeta_p(j)}.$$

In particular if we take $m = d$, we have $V_d^d = GL_d(\mathbb{Z}_p)$ and

$$\int_{GL_d(\mathbb{Z}_p)} dx = \prod_{1\le j\le d} \frac{1}{\zeta_p(j)}.$$

Normalizing the additive Haar measure dx by dividing by the above constant, one obtains the $GL_d(\mathbb{Z}_p)$ invariant probability measure on V_m^d. We denote by τ_m^d this measure on V_m^d, and $\tau^d := \tau_d^d$ the Haar measure on $GL_d(\mathbb{Z}_p)$.

Now we are interested in space

$$X_m^d := \mathrm{Grass}(m, d; \mathbb{Q}_p),$$

where $\mathrm{Grass}(m, d; \mathbb{Q}_p)$ is the Grassmann manifold of all m-dimensional space in d-dimensional plane over $\mathbb{Q}_p$. Note that $\mathrm{Grass}(m, d; \mathbb{Q}_p) = \mathrm{Grass}(m, d; \mathbb{Z}_p)$. Since $GL_d(\mathbb{Q}_p)$ (resp. $GL_d(\mathbb{Z}_p)$) acts transitively on X_m^d and the stabilizer of 1 is the Borel subgroup $B_{m,d-m}(\mathbb{Q}_p)$ (resp. $B_{m,d-m}(\mathbb{Z}_p)$) where

$$\begin{aligned} B_{m,d-m} :&= \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array} \right) \in \mathrm{Mat}_{d\times d} \;\middle|\; A \in GL_m,\ D \in GL_{d-m},\ B \in \mathrm{Mat}_{m\times(d-m)} \right\} \\ &= (GL_m \times GL_{d-m}) \ltimes \mathrm{Mat}_{m\times(d-m)}, \end{aligned}$$

we have

$$X_m^d = GL_d(\mathbb{Q}_p)/B_{m,d-m}(\mathbb{Q}_p) = GL_d(\mathbb{Z}_p)/B_{m,d-m}(\mathbb{Z}_p).$$

It can also be expressed as

$$X_m^d = \left\{ \mathfrak{p} \subseteq \mathbb{Z}_p^{\oplus d} \,\middle|\, \mathbb{Z}_p^{\oplus d}/\mathfrak{p} \simeq \mathbb{Z}_p^{\oplus (d-m)} \right\}.$$

Note that in the real case the factor $\mathrm{Mat}_{m\times(d-m)}(\mathbb{Z}_\eta)$ disappear, and the real Grassmann manifold resembles more the space

$$\begin{aligned} \widetilde{X}_m^d &= \left\{ (\mathfrak{p}, \mathfrak{p}') \,\middle|\, \mathfrak{p} \simeq \mathbb{Z}_p^{\oplus m},\ \mathfrak{p}' \simeq \mathbb{Z}_p^{\oplus(d-m)},\ \mathfrak{p} \oplus \mathfrak{p}' \simeq \mathbb{Z}_p^{\oplus d} \right\} \\ &= GL_d(\mathbb{Z}_p)/GL_m(\mathbb{Z}_p) \times GL_{d-m}(\mathbb{Z}_p). \end{aligned}$$

The measure $\overline{\tau}_m^d$ on X_m^d is obtained as follows; Let pr be the projection

$$\mathrm{pr} : V_m^d \longrightarrow X_m^d = V_m^d/GL_m(\mathbb{Z}_p)\,; \quad \mathrm{pr}(A_1, \ldots, A_m) = \mathrm{Span}_{\mathbb{Z}_p}(A_1, \ldots, A_m).$$

Then we see that the image $\mathrm{pr}_*(\tau_m^d)$ of the probability measure τ_m^d is the unique $GL_d(\mathbb{Z}_p)$ invariant probability measure on X_m^d. Hence, by the uniqueness, we have $\overline{\tau}_m^d = \mathrm{pr}_*(\tau_m^d)$. On the other hand, notice that the set of matrices $X \in \mathrm{Mat}_{d\times m}(\mathbb{Z}_p)$ of rank $X = m$ is of full measure with respect to the additive Haar measure dx. Then we have the projection

$$\widetilde{\mathrm{pr}} : \mathrm{Mat}_{d\times m}(\mathbb{Z}_p) \longrightarrow X_m^d = V_m^d/GL_m(\mathbb{Z}_p)\,; \quad \widetilde{\mathrm{pr}}(X) = \mathrm{Span}_{\mathbb{Q}_p}(X_1, \ldots, X_m) \cap \mathbb{Z}_p^{\oplus d}$$

and also $\overline{\tau}_m^d = \widetilde{\mathrm{pr}}_*(dx)$. Note that $\widetilde{\mathrm{pr}}(X)$ is not the space spanned by X over $\mathbb{Z}_p$.

The space X_m^d can be also represented as the inverse limit;

$$X_m^d = GL_d(\mathbb{Z}_p)/B_{m,d-m}(\mathbb{Z}_p) = \varprojlim X_{N^m}^{N^d},$$

where $X_{N^m}^{N^d}$ is the finite set defined by $X_{N^m}^{N^d} := GL_d(\mathbb{Z}/p^N)/B_{m,d-m}(\mathbb{Z}/p^N) \simeq G_{N^d}/B_{N^m}$ and $B_{N^m} := B_{m,d-m}(\mathbb{Z}/p^N)$. One can also check that G_{N^d} acts on $X_{N^m}^{N^d}$ transitively and the stabilizer of 1 is given by B_{N^m}.

9.1.2 Unitary Representations of $GL_d(\mathbb{Z}_p)$ and G_{N^d}

We are interested in the unitary representation of $GL_d(\mathbb{Z}_p)$ defined by

$$\pi : GL_d(\mathbb{Z}_p) \longrightarrow U(H_m^d)\,; \quad \pi(g)f(x) := f(g^{-1}x),$$

where $H_m^d := L^2(X_m^d, \overline{\tau}_m^d)$. Now the Hilbert space H_m^d can be written as the direct limit of the finite dimensional spaces as follows;

$$H_m^d = \varinjlim H_{N^m}^{N^d},$$

where $H_{N^m}^{N^d} := L^2(X_{N^m}^{N^d})$. We have a unitary embedding from the finite dimensional space $H_{N^m}^{N^d}$ to H_m^d and $\bigcup_N H_{N^m}^{N^d}$ is dense in H_m^d. Moreover, each finite dimensional space is invariant under the group $GL_d(\mathbb{Z}_p)$ and the representation of $GL_d(\mathbb{Z}_p)$ on it factors through the projection $GL_d(\mathbb{Z}_p) \twoheadrightarrow G_{N^d} \to U(H_{N^m}^{N^d})$. The commutant of this representation are generated by the Hecke algebra

$$\mathcal{H}_m^d := C^\infty(\Omega_m^d).$$

Notice that, in the p-adic cases, smoothness means locally constant. Here

$$\Omega_m^d := B_{m,d-m}(\mathbb{Z}_p)\backslash GL_d(\mathbb{Z}_p)/B_{m,d-m}(\mathbb{Z}_p) = \varprojlim \Omega_{N^m}^{N^d},$$

where $\Omega_{N^m}^{N^d} := B_{N^m}\backslash G_{N^d}/B_{N^m}$. The commutant of the representation of the finite group G_{N^d} on the finite dimensional space $H_{N^m}^{N^d}$ is also generated by the Hecke algebra

$$\mathcal{H}_{N^m}^{N^d} = C^\infty(\Omega_{N^m}^{N^d}).$$

Again $\mathcal{H}_m^d$ is expressed as the direct limit of the space $\mathcal{H}_{N^m}^{N^d}$;

$$\mathcal{H}_m^d = \varinjlim \mathcal{H}_{N^m}^{N^d}.$$

More generally, if we want the intertwining operator of the various representation for different m, say $H_{N^m}^{N^d} \to H_{N^n}^{N^d}$, we have to consider the module

$$\mathcal{H}_{N^m,N^n}^{N^d} := C^\infty(B_{N^m}\backslash G_{N^d}/B_{N^n}).$$

Notice that we always assume $m \le n \le \frac{1}{2}d$.

Now remember the simple facts for finite $\mathbb{Z}_p$-modules. Let $\mathfrak{m}$ be a finite $\mathbb{Z}_p$-module (resp. $\mathbb{Z}/p^N$-module). Then it is of the form of

$$\mathfrak{m} \simeq \bigoplus_i \mathbb{Z}/p^{\lambda_i} =: \mathbb{Z}/p^\lambda,$$

where $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l > 0)$ is a partition (resp. with $\lambda_1 \le N$). In this case, we say the type of $\mathfrak{m}$ is λ and write $\mathrm{typ}(\mathfrak{m}) = \lambda$. This is a complete isomorphism invariant. Namely, two modules are isomorphic if and only if they have the same type. (Note that all partitions are *decreasing.* All the people working in real or q-special functions use increasing partition while Macdonald use decreasing partition ([Mac]). Hence we have to change the notation unfortunately if we treat both the real and the p-adic cases.) We also use the following notation

$$(1^{r_1}, 2^{r_2}, \ldots, N^{r_N}) := (\underbrace{N, \ldots, N}_{d}, \ldots, \underbrace{1, \ldots, 1}_{r_1})$$

In particular, $(N^d) = (\underbrace{N, \ldots, N}_{d})$ and hence

$$\mathbb{Z}/p^{(N^d)} \simeq \left(\mathbb{Z}/p^N\right)^{\oplus d}.$$

This is why we use the notation G_{N^d}, which is the automorphism group of $\left(\mathbb{Z}/p^N\right)^{\oplus d}$. These are the highly symmetric modules. If we take a module $\mathfrak{m} \subset \left(\mathbb{Z}/p^N\right)^{\oplus d}$ of $\mathrm{typ}(\mathfrak{m}) = \lambda$, there exist a basis $X_1, \ldots, X_d$ for the free module $\left(\mathbb{Z}/p^N\right)^{\oplus d}$ such that $p^{N-\lambda_1} X_1, \ldots, p^{N-\lambda_d} X_d$ is the basis for $\mathfrak{m}$. Here $y_1, \ldots, y_l$ is the basis for $\mathfrak{m}$ of type λ means that (note that $\mathfrak{m}$ is not *free*) y_i's generate $\mathfrak{m}$ and of order exactly λ_i. Equivalently, every $m \in \mathfrak{m}$ can be uniquely written as $m = a_1 y_1 + \cdots + a_l y_l$ for some $a_i \in \mathbb{Z}/p^{\lambda_i}$. For example, given such a module $\mathfrak{m} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d}$ of $\mathrm{typ}(\mathfrak{m}) = \lambda$, we have

$$\mathrm{typ}\big(\left(\mathbb{Z}/p^N\right)^{\oplus d}/\mathfrak{m}\big) = (N - \lambda_d, \ldots, N - \lambda_1).$$

As a corollary of the elementary divisor, we have

Corollary 9.1.1. *Any isomorphism $g : \mathfrak{m} \to \mathfrak{m}'$ between two finite submodules $\mathfrak{m}, \mathfrak{m}' \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d}$ can be extended to $g \in \mathrm{Aut}\big(\left(\mathbb{Z}/p^N\right)^{\oplus d}\big) = G_{N^d}$.*

Therefore, the space of the relative positions $\Omega_{N^m}^{N^d}$ can be written as follows;

Corollary 9.1.2.

$$\Omega_{N^m}^{N^d} \simeq \left\{\lambda = (\lambda_1, \ldots, \lambda_l) \,\middle|\, \lambda_1 \leq N,\ \lambda'_1 \leq m\right\} =: \Lambda_{N^m},$$

where the isomorphism is given by

$$G_{N^d}(\mathfrak{m}_1, \mathfrak{m}_2) \longmapsto \mathrm{typ}(\mathfrak{m}_1 \cap \mathfrak{m}_2).$$

Here we denote by $\lambda' = (\lambda'_1 \ldots, \lambda'_n)$ the conjugate of λ defined by $\lambda'_j = \#\left\{i \,\middle|\, \lambda_i \geq j\right\}$.

Indeed, if for some $g \in G_{N^d}$ with $g(\mathfrak{m}_i) = \mathfrak{m}'_i$, then we have $\mathrm{typ}(\mathfrak{m}_1 \cap \mathfrak{m}_2) = \mathrm{typ}(\mathfrak{m}'_1 \cap \mathfrak{m}'_2)$. Conversely, if $\mathrm{typ}(\mathfrak{m}_1 \cap \mathfrak{m}_2) = \mathrm{typ}(\mathfrak{m}'_1 \cap \mathfrak{m}'_2)$, we have an isomorphism $g : \mathfrak{m}_1 \cap \mathfrak{m}_2 \to \mathfrak{m}'_1 \cap \mathfrak{m}'_2$. By Corollary 9.1.1, this can be extended to isomorphisms $g_i : \mathfrak{m}_i \to \mathfrak{m}'_i$ for $i = 1, 2$. Hence we have an isomorphism $g : \mathfrak{m}_1 + \mathfrak{m}_2 \to \mathfrak{m}'_1 + \mathfrak{m}'_2$. By Corollary 9.1.1 again, g can be extended to $g \in G_{N^d}$. This shows that $g(\mathfrak{m}_1, \mathfrak{m}_2) = (\mathfrak{m}'_1, \mathfrak{m}'_2)$.

Since $\mathrm{typ}(\mathfrak{m}_1 \cap \mathfrak{m}_2) = \mathrm{typ}(\mathfrak{m}_2 \cap \mathfrak{m}_1)$, we have the following

Corollary 9.1.3. *The Hecke algebra $\mathcal{H}^{N^d}_{N^m}$ is commutative. The dimension of $\mathcal{H}^{N^d}_{N^m}$ is given by $\#\Lambda_{N^m} = \binom{N+m}{m}$. Hence their direct limit $\mathcal{H}^d_m = \varinjlim \mathcal{H}^{N^d}_{N^m}$ is also commutative.*

Therefore the representations of G_{N^d} and $GL_d(\mathbb{Z}_p)$ are multiplicity free, whence they decompose as follows

$$H^{N^d}_{N^m} = \bigoplus_{\lambda \in \Lambda_{N^m}} V_\lambda, \qquad H^d_m = \bigoplus_{\lambda'_1 \leq m} V_\lambda.$$

We have the following diagrams using the quotient maps from modulo p^N to modulo p^{N-1}. Here the projection $\Lambda_{N^m} \to \Lambda_{(N-1)^m}$ is given by "chopping the right-most column", that is, $(\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_N) \mapsto (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{N-1})$;

$$\begin{array}{ccc} G_{N^d} & \longrightarrow & X^{N^d}_{N^m} \\ \downarrow & & \downarrow \\ G_{(N-1)^d} & \longrightarrow & X^{(N-1)^d}_{(N-1)^m} \end{array} \qquad \begin{array}{ccc} \Omega^{N^d}_{N^m} & \xrightarrow{\sim} & \Lambda_{N^m} \\ \downarrow & & \downarrow \\ \Omega^{(N-1)^d}_{(N-1)^m} & \longrightarrow & \Lambda_{(N-1)^m} \end{array}$$

Taking the inverse limit, we have the following trees;

N-th layer	tree	boundary
$X^{N^d}_{N^m}$	$\bigsqcup_N X^{N^d}_{N^m}$	$X^d_m = \varprojlim X^{N^d}_{N^m}$
Λ_{N^m}	$\bigsqcup_N \Lambda_{N^m}$	$\Omega^d_m = \varprojlim \Omega^{N^d}_{N^m}$

Notice that, we have infinite partitions in $\varprojlim \Lambda_{N^m} \simeq \varprojlim \Omega^{N^d}_{N^m}$, that is,

$$\varprojlim \Lambda_{N^m} = \Lambda_m \sqcup \Lambda_{m-1} \sqcup \cdots \sqcup \Lambda_1 \sqcup \Lambda_0 = \{\infty\},$$
$$\Lambda_{m-j} := \big\{\lambda = (\underbrace{\infty, \ldots, \infty}_{j} > \lambda_{j+1} \geq \ldots \geq \lambda_m \geq 0)\big\}.$$

We have the two types of embedding

$$\Omega^d_m = \varprojlim \Lambda_{N^m} \longrightarrow [0, 1]^m$$

defined as follows;

$$\begin{aligned}\text{sin-embedding } &: \lambda \longmapsto (p^{-\lambda_1}, \dots, p^{-\lambda_m}),\\ \text{cos-embedding } &: \lambda \longmapsto (1-p^{-\lambda_1}, \dots, 1-p^{-\lambda_m}).\end{aligned}$$

Here we understand $p^{-\infty} = 0$. It is important to note that we have two types of topologies in Ω_m^d, that is, the inverse limit topology, and the topology induced from $[0,1]^m$, and these are the same topology. This shows that the set of all finite partitions Λ_m (these do not have the 0 coordinate in $[0,1]^m$ by the embedding above) is an open and dense subspace of Ω_m^d; it is also of full measure with respect to the probability measure $\overline{\tau}_{m,n}^d$ on Ω_m^d. Here the measure $\overline{\tau}_{m,n}^d$ is obtained as follows; Let us write

$$\Omega_m^d = B_{m,d-m}(\mathbb{Z}_p)\backslash GL_d(\mathbb{Z}_p)/B_{n,d-n}(\mathbb{Z}_p)$$

Then $\overline{\tau}_{m,n}^d$ is the measure induced from the Haar measure τ^d on $GL_d(\mathbb{Z}_p)$. As in the case of the reals, $\overline{\tau}_{m,n}^d$ can be obtained by $t_*(dx \otimes dy)$. Here $dx \otimes dy$ is the additive measure on $\mathrm{Mat}_{d\times(m+n)}(\mathbb{Z}_p)$ and t is the map

$$t : \mathrm{Mat}_{d\times(m+n)}(\mathbb{Z}_p) \xrightarrow{\widetilde{\mathrm{pr}}} X_m^d \times X_n^d \xrightarrow{\mathrm{typ}} \Omega_m^d$$

It can be also expressed as

$$\overline{\tau}_{m,n}^d = t_*(dx \otimes dy) = t_*(dx \otimes \delta_{y_0}) = t_*(\delta_{x_0} \otimes dy)$$

for some $x_0 \in \mathrm{Mat}_{d\times m}(\mathbb{Z}_p)$, or some $y_0 \in \mathrm{Mat}_{d\times n}(\mathbb{Z}_p)$. We get the Markov chain on $\bigsqcup_N \Lambda_{N^m}$ with harmonic measure $\overline{\tau}_{m,n}^d$ (remember that we have the Markov chain if we have a tree and a measure on the boundary).

Now we try to see the relative position more like in the real case. Let $A, B \in \mathbb{P}^{d-1}(\mathbb{Z}_p^{\oplus d})$. Define

$$|(A,B)| = 1 - \rho(A,B) := \sup\bigl\{1 - p^{-n} \bigm| A \equiv B \pmod{p^n},\ n \ge 0\bigr\}.$$

For example, we have

$$\begin{aligned}A \not\equiv B \pmod{p} &\iff |(A,B)| = 1 - p^0 = 0\\ &\iff A, B \text{ are orthonormal,}\end{aligned}$$

and

$$\begin{aligned}A = B &\iff A \equiv B \pmod{p^n} \text{ for all } n \ge 0\\ &\iff |(A,B)| = 1.\end{aligned}$$

Hence we have for $\mathfrak{p} \in X_m^d$ and $\mathfrak{q} \in X_n^d$, $\mathrm{typ}(\mathfrak{p},\mathfrak{q}) = \lambda \in \varprojlim \Lambda_{N^m}$ if and only if there exists orthonormal basis $A_1, \dots, A_m$ for $\mathfrak{p}$ and $B_1, \dots, B_n$ for $\mathfrak{q}$ such that $|(A_i, B_j)| = \delta_{i,j}(1 - p^{-\lambda_i})$.

9.2 Harmonic Measure

9.2.1 Notations

Let $\lambda, \mu, \overline{\mu}$ be partitions. We put $G_\lambda := \mathrm{Aut}(\mathbb{Z}/p^\lambda)$ and fixing $\mathfrak{m}_0 = \mathbb{Z}/p^\lambda$ we define

$$\begin{aligned} X^\lambda_\mu &: = \mathrm{Grass}\big(\mathfrak{m} \subseteq \mathfrak{m}_0 \,\big|\, \mathrm{typ}(\mathfrak{m}) = \mu\big), \\ X^\lambda_{\mu,\overline{\mu}} &: = \mathrm{Grass}\big(\mathfrak{m} \subseteq \mathfrak{m}_0 \,\big|\, \mathrm{typ}(\mathfrak{m}) = \mu,\ \mathrm{typ}(\mathfrak{m}_0/\mathfrak{m}) = \overline{\mu}\big). \end{aligned}$$

More generally, for the modules $\mathfrak{m}_0$ of $\mathrm{typ}(\mathfrak{m}_0) = \lambda$ and $\mathfrak{m} \subseteq \mathfrak{m}_0$ of $\mathrm{typ}(\mathfrak{m}) = \mu$, we get the sequence of the partitions $\{\mathrm{typ}(\mathfrak{m}_0/\mathfrak{m} \cap p^i \mathfrak{m}_0)\}_{i=0,1,\dots}$ from $\overline{\mu}$ to λ. Hence

$$T := \big\{\mathrm{typ}(\mathfrak{m}_0/\mathfrak{m} \cap p^i \mathfrak{m}_0)\big\}_{i \geq 0}$$

is a tableau of shape $\mathrm{sh}(T) = \lambda \backslash \overline{\mu}$ and weight $\mathrm{wt}(T) = \mu$. We define for a given tableau T

$$X^\lambda_T : = \mathrm{Grass}\big(\mathfrak{m} \subseteq \mathfrak{m}_0 \,\big|\, \big\{\mathrm{typ}(\mathfrak{m}_0/\mathfrak{m} \cap p^i \mathfrak{m}_0)\big\}_{i \geq 0} = T\big).$$

Then the group G_{N^d} acts on the spaces X^λ_μ, $X^\lambda_{\mu,\overline{\mu}}$ and X^λ_T. Note that $X^\lambda_\mu = \bigcup_{\overline{\mu}} X^\lambda_{\mu,\overline{\mu}}$ and $X^\lambda_{\mu,\overline{\mu}} = \bigcup_T X^\lambda_T$ the union taken over T with $\mathrm{sh}(T) = \lambda \backslash \overline{\mu}$ and $\mathrm{wt}(T) = \mu$. G_{N^d} is not transitive on X^λ_T (It is very difficult combinatorial problem to describe all the equivalence classes of embedding $\mathbb{Z}/p^\mu \hookrightarrow \mathbb{Z}/p^\lambda$). We denote respectively by

$$\binom{\lambda}{T}_p := \# X^\lambda_T, \quad \binom{\lambda}{\mu, \overline{\mu}}_p := \# X^\lambda_{\mu,\overline{\mu}} = \sum_T \binom{\lambda}{T}_p, \quad \binom{\lambda}{\mu}_p := \# X^\lambda_\mu = \sum_{\overline{\mu}} \binom{\lambda}{\mu, \overline{\mu}}_p$$

$\binom{\lambda}{T}_p$ are monic polynomials in p, and $\binom{\lambda}{\mu,\overline{\mu}}_p$ are the Hall polynomial (see [Mac]). One can see that the leading term of $\binom{\lambda}{\mu,\overline{\mu}}_p$ is the number $c^\lambda_{\mu,\overline{\mu}}$ of tableau T with $\mathrm{sh}(T) = \lambda \backslash \overline{\mu}$ and $\mathrm{wt}(T) = \mu$; $c^\lambda_{\mu,\overline{\mu}}$ are the Littlewood-Richardson coefficients.

Let

$$[n]_p := \frac{1}{\zeta_p(n)} = 1 - p^{-n}, \quad [n]_p! := [n]_p \cdots [1]_p, \quad \begin{bmatrix} n \\ m \end{bmatrix}_p := \frac{[n]_p!}{[m]_p! [n-m]_p!}.$$

Then it is easy to see that

$$\begin{aligned} \#\mathrm{Hom}(\mathbb{Z}/p^\lambda, \mathbb{Z}/p^\mu) &= p^{\langle \lambda', \mu' \rangle}, \\ \#\mathrm{Hom}^{1:1}(\mathbb{Z}/p^\lambda, \mathbb{Z}/p^\mu) &= p^{\langle \lambda', \mu' \rangle} \prod_i \frac{[\mu'_i - \lambda'_{i+1}]_p!}{[\mu'_i - \lambda'_i]_p!}, \end{aligned}$$

where $\langle \lambda', \mu' \rangle := \sum_i \lambda'_i \mu'_i$. In particular, taking $\mu = \lambda$, we have

$$\#G_\lambda = p^{\langle \lambda', \lambda' \rangle} \prod_i [\lambda'_i - \lambda'_{i+1}]_p!.$$

Hence we have

$$\binom{\lambda}{\mu}_p = \#X^\lambda_\mu = \sum_{\overline{\mu}} \binom{\lambda}{\mu, \overline{\mu}}_p = \frac{\#\mathrm{Hom}^{1:1}(\mathbb{Z}/p^\mu, \mathbb{Z}/p^\lambda)}{\#G_\mu}$$
$$= p^{\langle \mu', \lambda' - \mu' \rangle} \prod_i \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_p.$$

Also we set

$$\{n\}_p := \frac{-1}{\zeta_p(-n)} = p^n - 1, \quad \{n\}_p! := \{n\}_p \cdots \{1\}_p, \quad \begin{Bmatrix} n \\ m \end{Bmatrix}_p := \frac{\{n\}_p!}{\{m\}_p!\{n-m\}_p!}.$$

These are useful notations when we count things. On the other hand we use the notation $[n]_p$ when we are working with the probability measure. Notice that

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_p = \begin{bmatrix} n \\ m \end{bmatrix}_p p^{m(n-m)}.$$

Then it can be calculated as

$$\binom{N^d}{N^m} = \#X^{N^d}_{N^m} = p^{Nm(d-m)} \begin{bmatrix} d \\ d-m \end{bmatrix}_p.$$

Similarly, for a general partition λ, it is useful to calculate

$$\frac{\begin{bmatrix} N^d \\ \lambda \end{bmatrix}_p}{\begin{bmatrix} (N-1)^d \\ \overline{\lambda} \end{bmatrix}_p} = \frac{\#X^{N^d}_\lambda}{\#X^{(N-1)^d}_{\overline{\lambda}}} = \frac{p^{\sum_{i=1}^N \lambda'_i(d-\lambda'_i)} \prod_{i=1}^N \begin{bmatrix} d-\lambda'_{i+1} \\ d-\lambda'_i \end{bmatrix}_p}{p^{\sum_{i=1}^{N-1} \lambda'_i(d-\lambda'_i)} \prod_{i=1}^{N-1} \begin{bmatrix} d-\overline{\lambda}'_{i+1} \\ d-\overline{\lambda}'_i \end{bmatrix}_p}$$
$$= p^{\lambda'_N(d-\lambda'_N)} \frac{\begin{bmatrix} d \\ d-\lambda'_N \end{bmatrix}_p \begin{bmatrix} d-\lambda'_N \\ d-\lambda'_{N-1} \end{bmatrix}_p}{\begin{bmatrix} d \\ d-\lambda'_{N-1} \end{bmatrix}_p}.$$

Here $\overline{\lambda}$ is the projection of λ; $\overline{\lambda'} = (\lambda'_1, \ldots, \lambda'_{N-1})$.

9.2.2 Harmonic Measure on Ω^d_m

Now we determine the harmonic measure $\tau := \overline{\tau}^d_{m,n}$ on the boundary space $\Omega^d_m = \varprojlim \Lambda_{N^m}$ of the relative positions of m-plane and n-plane from the transition probability of the Markov chain (see Sect. 9.2). We here work with the conjugate coordinate, that is, $\lambda = (\lambda'_1, \ldots, \lambda'_N)$ and $\overline{\lambda} = (\lambda'_1, \ldots, \lambda'_{N-1})$.

Let τ_N be the probability measure on Λ_{N^m}. First of all, let us calculate the measure in the finite layer $\tau_1(\lambda_1')$. Fix a subspace $\mathfrak{q}_1 = \mathbb{F}_p^{\lambda_1'}$ with $\mathfrak{q}_1 \subseteq \mathfrak{q}_0 = \mathbb{F}_p^n \subseteq \mathbb{F}_p^d$. Note that

$$\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m,\ \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1\} = \begin{Bmatrix} d-n \\ m-\lambda_1' \end{Bmatrix}_p p^{(m-\lambda_1')(n-\lambda_1')},$$

$$\#\{\mathfrak{q}_1 \subseteq \mathbb{F}_p^n \,|\, \dim\mathfrak{q}_1 = \lambda_1'\} = \begin{Bmatrix} n \\ \lambda_1' \end{Bmatrix}_p,$$

$$\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m\} = \begin{Bmatrix} d \\ m \end{Bmatrix}_p.$$

Hence the first transition probability of the Markov chain is calculated as

$$\begin{aligned}
\tau_1(\lambda_1') &= \frac{\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m,\ \dim\mathfrak{p} \cap \mathfrak{q}_0 = \lambda_1'\}}{\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m\}} \\
&= \frac{\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m,\ \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1\} \cdot \#\{\mathfrak{q}_1 \subseteq \mathbb{F}_p^n \,|\, \dim\mathfrak{q}_1 = \lambda_1'\}}{\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m\}} \\
&= \frac{\left\{\begin{smallmatrix} d-n \\ m-\lambda_1' \end{smallmatrix}\right\}_p p^{(m-\lambda_1')(n-\lambda_1')} \cdot \left\{\begin{smallmatrix} n \\ \lambda_1' \end{smallmatrix}\right\}_p}{\left\{\begin{smallmatrix} d \\ m \end{smallmatrix}\right\}_p} \\
&= \frac{\left[\begin{smallmatrix} d-n \\ m-\lambda_1' \end{smallmatrix}\right]_p \left[\begin{smallmatrix} n \\ \lambda_1' \end{smallmatrix}\right]_p}{\left[\begin{smallmatrix} d \\ m \end{smallmatrix}\right]_p} p^{-\lambda_1'(d-n-m+\lambda_1')}. \qquad (9.1)
\end{aligned}$$

Next we work on the N-th layer. For details see [On1]. Fix also a subspace $\mathfrak{q}_0 = \left(\mathbb{Z}/p^N\right)^{\oplus n} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d}$. Then we have

$$\begin{aligned}
\frac{\tau_N(\lambda)}{\tau_{N-1}(\overline{\lambda})} &= \frac{\binom{N^d}{N^m}_p^{-1} \#\{\mathfrak{p} \subseteq (\mathbb{Z}/p^N)^{\oplus d} \,|\, \mathrm{typ}(\mathfrak{p}) = N^m,\ \mathrm{typ}(\mathfrak{p} \cap \mathfrak{q}_0) = \lambda\}}{\binom{(N-1)^d}{(N-1)^m}_p^{-1} \#\{\overline{\mathfrak{p}} \subseteq (\mathbb{Z}/p^{N-1})^{\oplus d}, \,|\, \mathrm{typ}(\overline{\mathfrak{p}}) = (N-1)^m,\ \mathrm{typ}(\overline{\mathfrak{p}} \cap \overline{\mathfrak{q}_0}) = \overline{\lambda}\}} \\
&= \frac{\binom{N^d}{N^m}_p^{-1} \binom{N^n}{\lambda}_p \#\{\mathfrak{p} \subseteq (\mathbb{Z}/p^N)^{\oplus d} \,|\, \mathrm{typ}(\mathfrak{p}) = N^m,\ \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1\}}{\binom{(N-1)^d}{(N-1)^m}_p^{-1} \binom{(N-1)^n}{\overline{\lambda}}_p \#\{\overline{\mathfrak{p}} \subseteq (\mathbb{Z}/p^{N-1})^{\oplus d} \,|\, \mathrm{typ}(\overline{\mathfrak{p}}) = (N-1)^m,\ \overline{\mathfrak{p}} \cap \overline{\mathfrak{q}_0} = \overline{\mathfrak{q}_1}\}}
\end{aligned}$$

Here we fix $\mathfrak{q}_1$ of $\mathrm{typ}(\mathfrak{q}_1) = \lambda$. The independence of this choice of $\mathfrak{q}_1$ is justified by the symmetry of $\mathfrak{q}_0$. Hence we have

$$\begin{aligned}
\frac{\tau_N(\lambda)}{\tau_{N-1}(\overline{\lambda})} &= p^{-m(d-m)} \begin{bmatrix} \lambda_{N-1}' \\ \lambda_N' \end{bmatrix}_p p^{\lambda_N'(n-\lambda_N')} \\
&\times \#\{\mathfrak{p} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d} \,|\, \mathrm{typ}(\mathfrak{p}) = N^m,\, \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1,\, \overline{\mathfrak{p}} = \overline{\mathfrak{p}_0}\}. \qquad (9.2)
\end{aligned}$$

Here we again fix the submodule $\overline{\mathfrak{p}_0} \subseteq \left(\mathbb{Z}/p^{N-1}\right)^{\oplus d}$ of $\mathrm{typ}(\overline{\mathfrak{p}_0}) = (N-1)^m$ such that $\overline{\mathfrak{p}_0} \cap \overline{\mathfrak{q}_0} = \overline{\mathfrak{q}_1}$. To calculate (9.2), without loss of generality, we assume that

$$\lambda'_N = 0 \tag{9.3}$$

since we can factor out $\left(\mathbb{Z}/p^N\right)^{\oplus \lambda'_N} \subseteq \mathfrak{p} \cap \mathfrak{q}$. Hence, to calculate (9.2), it is sufficient to count

$$\#\{\mathfrak{p} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus (d-\lambda'_N)} \,\big|\, \mathrm{typ}(\mathfrak{p}) = N^{m-\lambda'_N},\ \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1,\ \overline{\mathfrak{p}} = \overline{\mathfrak{p}_0}\}. \tag{9.4}$$

Fix $\mathfrak{q}_0 \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus n-\lambda'_N}$, $\mathfrak{q}_1$ of $\mathrm{typ}(\mathfrak{q}_1) = \overline{\lambda}$ and $\overline{\mathfrak{p}_0}$ of $\mathrm{typ}(\overline{\mathfrak{p}_0}) = (N-1)^{m-\lambda'_N}$. Also fix a lifting $\mathfrak{p}_0$ of $\overline{\mathfrak{p}_0}$. Let $\mathfrak{A} := \{A_1, \ldots, A_m\}$ be a basis for $\mathfrak{p}_0$ and $\mathfrak{B} := \{B_1, \ldots, B_m\}$ a completion to a basis for $\left(\mathbb{Z}/p^N\right)^{\oplus d}$. Then any other lifting $\mathfrak{p}$ of $\overline{\mathfrak{p}_0}$ has basis of the form

$$A_i + p^{N-1}\{\sum_{1 \le j \le m} a_{ij} A_j + \sum_{1 \le k \le m} b_{ik} B_k\},$$

where $a_{ij}, b_{ik} \in \mathbb{F}_p$. Now note that a_{ij}'s do not change $\mathfrak{p}$. Hence we ignore a_{ij}'s and consider only b_{ik}'s. Note also that, for two liftings $\mathfrak{p}$ and $\mathfrak{p}'$ of $\overline{\mathfrak{p}_0}$ given by $\{b_{ik}\}$ and $\{b'_{ik}\}$ respectively, it holds that $\mathfrak{p} = \mathfrak{p}'$ if and only if $b_{ik} = b'_{ik}$. Therefore the choice of the $\{b_{ik}\}$ determines the space uniquely. Now let $\mathfrak{C} = \{C_1, \ldots, C_n\}$ be the basis for $\mathfrak{q}_0$ such that $p^{N-\lambda_i} C_i = p^{N-\lambda_i} A_i$ is a basis for $\mathfrak{q}_0 \cap \mathfrak{p}_0 = \mathfrak{q}_1 = \mathbb{Z}/p^\lambda$. Write

$$\mathfrak{A} = \bigsqcup_{0 \le k \le N} \mathfrak{A}^{(k)}, \qquad \mathfrak{C} = \bigsqcup_{0 \le k \le N} \mathfrak{C}^{(k)}$$

with $p^{N-k}\mathfrak{C}^{(k)} = p^{N-k}\mathfrak{A}^{(k)}$. Hence we have $\#\mathfrak{C}^{(k)} = \#\mathfrak{A}^{(k)} = \#\{i \,|\, \lambda_i = k\}$. By the assumption (9.3), $\#\mathfrak{C}^{(N)} = \mathfrak{A}^{(N)} = 0$. Note that the element of $\bigsqcup_{0 \le k < N-1} \mathfrak{A}^{(k)}$ can be changed arbitrary and the number of such choices (i.e., the choices of $\{b_{ij}\}$'s) are equal to $p^{(d-m)(n-\lambda'_{N-1})}$. On the other hand the element of $\mathfrak{A}^{(N-1)} = \{A_{m-\lambda'_{N-1}+1}, \ldots, A_m\}$ cannot be changed arbitrary, only by b_{ij}'s, which avoid $p^{N-1}(\mathfrak{C} \backslash \mathfrak{C}^{(N-1)})$ (notice that $\dim_{\mathbb{F}_p}(\mathfrak{C} \backslash \mathfrak{C}^{(N-1)}) = n - \lambda'_{N-1}$). Hence when we chose $\{b_{ij}\}$'s, we have to avoid not only the space $\mathfrak{C} \backslash \mathfrak{C}^{(N-1)}$ but also the space spanned by $(i-1)$ elements chosen previously. Because we fix $\mathfrak{p}_0 \cap \mathfrak{q}_1 = \mathfrak{q}_0$, the number of choices of such elements is $p^{d-m} - p^{n-\lambda'_{N-1}+i-1}$. Therefore all the number (9.4) is given by

$$\begin{aligned}
&p^{(d-m)(m-\lambda'_{N-1})}(p^{d-m} - p^{n-\lambda'_{N-1}})(p^{d-m} - p^{n-\lambda'_{N-1}+1}) \cdots \\
&\qquad (p^{d-m} - p^{n-\lambda'_{N-1}+\lambda'_{N-1}-1}) \\
&= p^{(d-m)m}(1 - p^{-(d-m-n+\lambda'_{N-1})}) \cdots (1 - p^{-(d-m-n+1)}) \\
&= p^{(d-m)m} \frac{[d-m-n+\lambda'_{N-1}]_p!}{[d-m-n]_p!}.
\end{aligned}$$

To remove the assumption (9.3), we substitute $d - \lambda'_N$ for d, and so on, i.e., subtract λ'_N from d, m, n and λ'_{N-1}. Then we have

$$\begin{aligned} &\#\{\mathfrak{p} \subseteq (\mathbb{Z}/p^N)^{\oplus d} \,|\, \mathrm{typ}(\mathfrak{p}) = N^m,\, \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1,\, \overline{\mathfrak{p}} = \overline{\mathfrak{p}_0}\} \\ &\qquad = \frac{[d-m-n+\lambda'_{N-1}]_p!}{[d-m-n+\lambda'_N]_p!} p^{(d-m)(m-\lambda'_N)}. \end{aligned} \tag{9.5}$$

Hence, from (9.2) and (9.5), the transition probability is given by

$$\begin{aligned} \frac{\tau_N(\lambda)}{\tau_{N-1}(\overline{\lambda})} &= p^{-m(d-m)} \begin{bmatrix} \lambda'_{N-1} \\ \lambda'_N \end{bmatrix}_p p^{\lambda'_N(n-\lambda'_N)} \frac{[d-m-n+\lambda'_{N-1}]_p!}{[d-m-n+\lambda'_N]_p!} p^{(d-m)(m-\lambda'_N)} \\ &= \begin{bmatrix} \lambda'_{N-1} \\ \lambda'_N \end{bmatrix}_p \frac{[d-m-n+\lambda'_{N-1}]_p!}{[d-m-n+\lambda'_N]_p!} p^{-\lambda'_N(d-m-n+\lambda'_N)}. \end{aligned}$$

Therefore, taking the product of all $0 \le j \le N$ of the transition probability, the measure τ_N on the N-th layer is calculated as follows;

$$\begin{aligned} \tau_N(\lambda) &= \frac{\begin{bmatrix} n \\ \lambda'_1 \end{bmatrix}_p \begin{bmatrix} d-n \\ m-\lambda'_1 \end{bmatrix}_p}{\begin{bmatrix} d \\ m \end{bmatrix}_p} p^{-\lambda'_1(d-n-m+\lambda'_1)} \\ &\quad \prod_{1 \le j \le N} \begin{bmatrix} \lambda'_{j-1} \\ \lambda'_j \end{bmatrix}_p \frac{[d-m-n+\lambda'_{j-1}]_p!}{[d-m-n+\lambda'_j]_p!} p^{-\lambda'_j(d-m-n+\lambda'_j)} \\ &= \begin{bmatrix} n \\ n-\lambda'_1, \lambda'_1-\lambda'_2, \ldots, \lambda'_N \end{bmatrix}_p \\ &\quad \frac{\frac{[d-n]_p!}{[m-\lambda'_1]_p![d-m-n+\lambda_N]_p!} p^{-\sum_{i=1}^N \lambda'_i(d-m-n+\lambda'_i)}}{\begin{bmatrix} d \\ m \end{bmatrix}_p}, \end{aligned}$$

where

$$\begin{bmatrix} n \\ m_1, \ldots, m_N \end{bmatrix}_p := \frac{[n]_p!}{[m_1]_p! \cdots [m_N]_p!} \qquad (m_1 + \cdots + m_N = n)$$

is the multinomial coefficient. Then the harmonic measure τ is obtained by taking the limit $N \to \infty$ of the measure τ_N on the N-th layer;

$$\tau(\lambda) := \begin{bmatrix} n \\ n-\lambda'_1, \lambda'_1-\lambda'_2, \ldots \end{bmatrix}_p \frac{\frac{[d-n]_p!}{[m-\lambda'_1]_p![d-m-n]_p!} p^{-\sum_{i \ge 1} \lambda'_i(d-m-n+\lambda'_i)}}{\begin{bmatrix} d \\ m \end{bmatrix}_p}.$$

It can be written as follows

$$\tau(\lambda) = \frac{\begin{bmatrix} d \\ m+n \end{bmatrix}_p}{\begin{bmatrix} d \\ m \end{bmatrix}_p \begin{bmatrix} d \\ n \end{bmatrix}_p} \frac{[m+n]!}{[m-\lambda'_1]_p![n-\lambda'_1]_p!} \prod_{j \ge 1} \frac{1}{[\lambda'_j - \lambda'_{j+1}]_p!} p^{-\sum_{i \ge 1} \lambda_i(d-m-n+2i-1)}. \tag{9.6}$$

This expression shows that $\tau(\lambda)$ is symmetric in m and n. We call this measure the harmonic Selberg measure, which is a p-adic analogue of the Selberg measure.

9.3 Basis for the Hecke Algebra

In the last section, we see the unitary representation of $G_{N^d} = GL_d(\mathbb{Z}/p^N)$; $\pi : G_{N^d} \to U(H_{N^m}^{N^d})$ where $H_{N^m}^{N^d} := L^2(X_{N^m}^{N^d}, \tau)$. The commutant is generated by the Hecke algebra $\mathcal{H}_{N^m}^{N^d} = L^2(\Lambda_{N^m})$. We have the geometric basis $\{\delta_\lambda\}_{\lambda \subseteq N^m}$ for $\mathcal{H}_{N^m}^{N^d}$, which act on the function in $H_{N^m}^{N^d}$ as

$$\delta_\lambda \varphi(y) := \int_{\mathrm{typ}(x \cap y) = \lambda} \varphi(x)\tau(x) \qquad (\varphi \in H_{N^m}^{N^d}).$$

On the other hand, we denote by $\ell^2(X_{N^m}^{N^d})$ the Hilbert space with the counting measure (not normalized to be a probability measure). In this case, we denote by g_λ the geometric basis, acting via

$$g_\lambda \varphi(y) := \sum_{\mathrm{typ}(x \cap y) = \lambda} \varphi(x) \qquad (\varphi \in H_{N^m}^{N^d}).$$

Note that g_λ is up to constant identical with δ_λ, that is,

$$g_\lambda = \binom{N^d}{N^m}_p \delta_\lambda.$$

Let $\lambda' \subseteq \lambda \subseteq N^d$. We define "gradient" and "divergent" operators

$$\ell^2(X_\lambda^{N^d}) \underset{T_{\lambda \supseteq \lambda'}}{\overset{T_{\lambda' \subseteq \lambda}}{\rightleftarrows}} \ell^2(X_{\lambda'}^{N^d})$$

by

$$T_{\lambda' \subseteq \lambda} \varphi(x') := \sum_{x' \subseteq x} \varphi(x), \qquad T_{\lambda \supseteq \lambda'} \varphi(x) := \sum_{x \supseteq x'} \varphi(x').$$

It is clear that these operators are adjoint to each other and commute with the action of G_{N^d} on the Grassmann manifolds. Let $\lambda_1, \lambda_2 \subseteq \lambda \subseteq N^d$. Then we also define

$$T_{\lambda_1,\lambda_2}^{\lambda} : \ell^2(X_{\lambda_2}^{N^d}) \longrightarrow \ell^2(X_{\lambda_1}^{N^d})$$

by

$$T_{\lambda_1,\lambda_2}^{\lambda} \varphi(x_1) = \sum_{\mathrm{typ}(x_1 + x_2) = \lambda} \varphi(x_2).$$

Then we have $(T_{\lambda_1,\lambda_2}^{\lambda})^* = T_{\lambda_2,\lambda_1}^{\lambda}$. This also commutes with G_{N^d}-action.

The "Laplacian" $c_\lambda : \ell^2(X^{N^d}_{N^m}) \to \ell^2(X^{N^d}_{N^m})$ is expressed in terms of the geometric basis:

$$c_\lambda = T_{N^m \supseteq \lambda} \circ T_{\lambda \subseteq N^m} = \sum_{\lambda \subseteq \lambda' \subseteq N^m} \binom{\lambda'}{\lambda}_p g_{\lambda'}.$$

The collection $\{c_\lambda\}_{\lambda \subseteq N^m}$ is called the cellular basis for $\mathcal{H}^{N^d}_{N^m}$. Note that the matrix $\{\binom{\lambda'}{\lambda}_p\}_{\lambda,\lambda'}$, which transforms the geometric basis $\{g_\lambda\}$ to the cellular basis $\{c_\lambda\}$, is upper triangular with $\binom{\lambda}{\lambda}_p = 1$. Let $\{\binom{\lambda'}{\lambda}^*_p\}_{\lambda,\lambda'} := \{\binom{\lambda'}{\lambda}_p\}^{-1}_{\lambda,\lambda'}$ denote the coefficients of inverse matrix. Then we have

$$g_\lambda = \sum_{p^{\lambda'} \subseteq \lambda \subseteq \lambda' \subseteq N^m} \binom{\lambda'}{\lambda}^*_p c_{\lambda'}$$

Moreover, we have explicit expression of $\binom{\beta}{\lambda}^*_p$;

$$\binom{\beta}{\lambda}^*_p := (-1)^{|\beta|-|\lambda|} p^{n(\beta)-n(\lambda)} \prod_{1 \le i \le m} \begin{bmatrix} \beta'_i - \beta'_{i+1} \\ \beta'_i - \lambda'_i \end{bmatrix}_p,$$

where $|\lambda| := \sum_i \lambda_i$ and $n(\lambda) := \sum_i \lambda_i(i-1)$ (these are the standard notations for partitions). Remark that

$$\mathcal{H}^{N^d}_{N^m}(\lambda) := \mathrm{Span}\{c_\alpha \mid \alpha \subseteq \lambda\}, \qquad \mathcal{H}^{N^d}_{N^m}(\lambda^-) := \mathrm{Span}\{c_\alpha \mid \alpha \subsetneq \lambda\}$$

are ideals of $\mathcal{H}^{N^d}_{N^m}$. Let us consider the quotient

$$\mathcal{W}_\lambda := \mathcal{H}^{N^d}_{N^m}(\lambda)/\mathcal{H}^{N^d}_{N^m}(\lambda^-) \qquad (\lambda \in \Lambda_m).$$

Then $\{\mathcal{W}_\lambda\}_{\lambda \in \Lambda_m}$ gives the complete list of the irreducible representations of the Hecke algebra $\mathcal{H}^{N^d}_{N^m}$. Note that $\dim \mathcal{W}_\lambda = 1$. Hence we have idempotents $\{\varphi_\lambda\}_{\lambda \in \Lambda_{N^m}}$ for $\mathcal{H}^{N^d}_{N^m}$:

$$\mathcal{W}_\lambda = \mathbb{C} \cdot \varphi_\lambda = \mathcal{H}^{N^d}_{N^m} * \varphi_\lambda.$$

The function φ_λ is characterized as

$$\begin{aligned} c_\alpha \cdot \varphi_\lambda &= 0 \quad \text{for } \alpha \subsetneq \lambda, \\ c_\lambda \cdot \varphi_\lambda &\neq 0. \end{aligned}$$

Then, the irreducible decomposition of $\ell^2(X^{N^d}_{N^m})$ is given by

$$\ell^2(X^{N^d}_{N^m}) = \bigoplus_{\lambda \in \Lambda_m} \mathcal{V}_\lambda,$$

we have also $\mathcal{V}_\lambda = \ell^2(X^{N^d}_{N^m}) * \varphi_\lambda$. Notice that $\mathcal{V}_\lambda$ is the unique irreducible representation of G_{N^d} which occurs in the Grassmann manifold $X^{N^d}_\lambda$ but does not occur in $X^{N^d}_\alpha$ for any $\alpha \subsetneq \lambda$.

Let us write

$$c_\lambda = \sum_{\alpha \subseteq \lambda} A_{\lambda,\alpha} \varphi_\alpha, \qquad \varphi_\lambda = \sum_{\alpha \subseteq \lambda} A^*_{\lambda,\alpha} c_\alpha$$

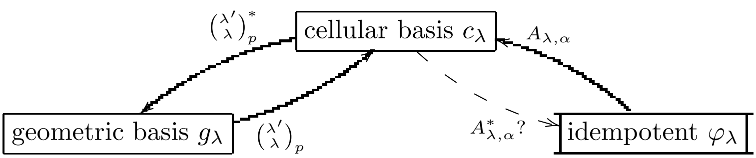
with some coefficients $A_{\lambda,\alpha}$ and $A^*_{\lambda,\alpha}$. Then the matrix $\{A_{\lambda,\alpha}\}_{\lambda,\alpha}$ is lower triangular and $\{A^*_{\lambda,\alpha}\}_{\lambda,\alpha} = \{A_{\lambda,\alpha}\}^{-1}_{\lambda,\alpha}$. Moreover, let

$$c_{\lambda_1} * c_{\lambda_2} = \sum_{\alpha \subseteq \lambda_1,\lambda_2} C^{\lambda_1,\lambda_2}_\alpha \cdot c_\alpha.$$

Then we have $A_{\lambda,\alpha} = C^{\lambda,\alpha}_\alpha$ for $\alpha \subseteq \lambda$. One can explicitly calculate the number $A_{\lambda,\alpha}$ as follows: Fixed submodules $\mathbb{Z}/p^\alpha \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus m}$ and $\mathbb{Z}/p^\lambda, \mathbb{Z}/p^\alpha \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d}$ such that $\mathbb{Z}/p^\lambda \cap \mathbb{Z}/p^\alpha = 0$. Then we have

$$\begin{aligned} A_{\lambda,\alpha} &= \#\{\mathfrak{m} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus m} \mid \mathbb{Z}/p^\alpha \subseteq \mathfrak{m},\ \mathrm{typ}(\mathfrak{m}) = \lambda\} \\ &\qquad \times \#\{\mathfrak{m} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d} \mid \mathbb{Z}/p^\lambda \subseteq \mathfrak{m},\ \mathfrak{m} \cap \mathbb{Z}/p^\alpha = 0,\ \mathrm{typ}(\mathfrak{m}) = N^m\} \binom{N^d}{N^m}_p^{-1} \\ &= p^{-(d-2m)|\lambda| - m|\alpha| - \langle \lambda', \lambda' - \alpha' \rangle} \begin{bmatrix} m - \alpha'_1 \\ m - \lambda'_1 \end{bmatrix}_p \prod_{i \geq 1} \begin{bmatrix} \lambda'_i - \alpha'_{i+1} \\ \lambda'_i - \lambda'_{i+1} \end{bmatrix}_p \begin{bmatrix} d - \lambda'_1 - \alpha'_1 \\ m - \lambda'_1 \end{bmatrix}_p \begin{bmatrix} d \\ m \end{bmatrix}_p^{-1}. \end{aligned}$$

It seems that the inverse matrix $\{A^*_{\lambda,\alpha}\}_{\lambda,\alpha}$ should be also calculated explicitly, however, unfortunately, we can not obtain this (it should be possible). For the reference of this section, see [BO1].



10

q-Grassmann Manifold

Summary. In Chap. 10 we describe briefly the quantum Grassmannian, the q-Selberg measure, and the multivariable (Little) q-Jacobi polynomials which are the idempotents and interpolate between the p-adic and the real analogues.

10.1 q-Selberg Measures

In this section we describe the q-theory for the Selberg measure which interpolates between the p-adic and the real one. The q-Selberg measure $S_q(\lambda) = S_q^{\alpha,\beta,\gamma}(\lambda)$ with parameters α, β and γ is defined by

$$S_q(\lambda) := \frac{1}{C_q^{\alpha,\beta,\gamma}} \prod_{1\le j\le m} \frac{\zeta_q(\alpha+\lambda_j+(m-j)\gamma)}{\zeta_q(1+\lambda_j+(m-j)\gamma)} q^{\lambda_j(\beta+2\gamma(j-1))} \cdot \Delta_q^\gamma(\lambda) \qquad (\lambda \in \Lambda_m). \tag{10.1}$$

Here $C_q^{\alpha,\beta,\gamma}$ is the normalization constant so that $S_q(\lambda)$ becomes a probability measure; it is the higher rank beta function;

$$C_q^{\alpha,\beta,\gamma} := \prod_{1\le j\le m} \frac{\zeta_q(\alpha+\gamma(j-1))\zeta_q(\beta+\gamma(m-j))\zeta_q(j\gamma)}{\zeta_q(\alpha+\beta+\gamma(m+j-2))\zeta_q(\gamma)\zeta_q(1)}. \tag{10.2}$$

$\Delta_q^\gamma(\lambda)$ is the Vandermonde determinant defined by

$$\Delta_q^\gamma(\lambda) := \prod_{j<i\le m} \frac{\zeta_q(\lambda_j-\lambda_i+\gamma(1+i-j))}{\zeta_q(1+\lambda_j-\lambda_i+\gamma(i-j-1))} (1-q^{\lambda_j-\lambda_i+\gamma(i-j)}).$$

Note again that partitions are decreasing. It was conjectured by Askey [As1] and proved by Kadell [Kad] and Habsieger [Hab] that $S_q(\lambda)$ is a probability measure on Λ_m when $\gamma \in \mathbb{N}$. Further, Aomoto [Ao] and Kaneko [Kan] prove this in the case where γ is a continuous parameter.

10.1.1 The p-Adic Limit of the q-Selberg Measures

We first consider the p-adic limit Ⓟ of $S_q(\lambda)$. Remember that, in the p-adic limit, we take $q \to 0$, $\alpha, \beta, \gamma \to 0$ in such a way that $q^\alpha \to p^{-\alpha}$, $q^\beta \to p^{-\beta}$ and $q^\gamma \to p^{-\gamma}$. Then we have

$$S_q(\lambda) \to S_p(\lambda) := \frac{1}{C_p^{\alpha,\beta,\gamma}} \prod_{\substack{1\le j\le m\\ \lambda_j=0}} \zeta_p(\alpha+(m-j)\gamma)p^{-\sum_j \lambda_j(\beta+2\gamma(j-1))} \cdot \Delta_p^\gamma(\lambda),$$

where

$$C_p^{\alpha,\beta,\gamma} := \prod_{1\le j\le m} \frac{\zeta_p(\alpha+\gamma(j-1))\zeta_p(\beta+\gamma(m-j))\zeta_p(j\gamma)}{\zeta_p(\alpha+\beta+\gamma(m+j-2))\zeta_p(\gamma)},$$

$$\Delta_q^\gamma(\lambda) := \prod_{\substack{j<i\le m\\ \lambda_j=\lambda_i}} \frac{\zeta_p(\gamma(1+i-j))}{\zeta_p(\gamma(i-j))}.$$

Let $\alpha = 1+n-m$, $\beta = 1+d-m-n$ and $\gamma = 1$. Then, noting

$$\lambda = (\ldots, 2, \ldots, \underset{\underset{\lambda_2'}{\uparrow}}{2}, 1, \ldots, \underset{\underset{\lambda_1'}{\uparrow}}{1}, 0, \ldots, \underset{\underset{m=\lambda_0'}{\uparrow}}{0}\ ),$$

$\sum_j \lambda_j = \sum_j \lambda_j'$ and $\sum_j \lambda_j(2j-1) = \sum_j (\lambda_j')^2$, we have

$$\begin{aligned} S_p(\lambda) = &\prod_{1\le j\le m} \frac{\zeta_p(d-m+j)\zeta_p(1)}{\zeta_p(n-m+j)\zeta_p(1+d-n-j)\zeta_p(j)} \\ &\times \prod_{\substack{j\\ \lambda_j=0}} \zeta_p(1+n-j)\cdot p^{-\sum_j \lambda_j(d-m-n+2j-1)} \prod_{\substack{j<i\\ \lambda_j=\lambda_i}} \frac{\zeta_p(1+i-j)}{\zeta_p(i-j)} \\ = &\frac{[n]_p!}{[n-m]_p!}\frac{[d-n]_p!}{[d-n-m]_p!}\frac{[d-m]_p!}{[d]_p!}\frac{[m]_p!}{[1]_p^m} \\ &\times \prod_{\lambda_1'<j\le m} \zeta_p(1+n-j)\cdot p^{-\sum_j \lambda_j'(d-m-n+\lambda_j')} \prod_{k\ge 0}\prod_{\lambda_{k+1}'<j<i\le\lambda_k'} \frac{\zeta_p(1+i-j)}{\zeta_p(i-j)}. \end{aligned} \tag{10.3}$$

Here we have

$$\prod_{\lambda_1'<j\le m} \zeta_p(1+n-j) = \frac{[n-m]_p!}{[n-\lambda_1']_p!}. \tag{10.4}$$

On the other hand, consider the set $\{(i,j) \mid \lambda_{k+1}' < j < i \le \lambda_k'\}$. We have the decomposition of this set into disjoint unions

$$\begin{aligned} \{(i,j) \mid \lambda_{k+1}' < j < i \le \lambda_k'\} &= \{(i,j) \mid j+1 < i\} \sqcup \{(i,j) \mid j+1 = i\} \\ &= \{(i,j) \mid \lambda_{k+1}'+1 < j\} \sqcup \{(i,j) \mid j_{k+1}'+1 = j\} \end{aligned}$$

and the bijection

$$s:\{(i,j)\,|\,j+1<i\}\xrightarrow{\simeq}\{(i,j)\,|\,\lambda'_{k+1}+1<j\};\quad (i,j)\longmapsto(i,j-1).$$

Therefore we get

$$\begin{aligned}\prod_{k\geq 0}\prod_{\lambda'_{k+1}<j<i\leq\lambda'_k}\frac{\zeta_p(1+i-j)}{\zeta_p(i-j)}&=\prod_{k\geq 0}\prod_{1+\lambda'_{k+1}<i\leq\lambda'_k}\frac{\zeta_p(i-\lambda'_{k+1})}{\zeta_p(1)}\\&=\prod_{k\geq 0}\prod_{\lambda'_{k+1}<i\leq\lambda'_k}\frac{\zeta_p(i-\lambda'_{k+1})}{\zeta_p(1)}\\&=\frac{[1]_p^m}{[m-\lambda_1]_p![\lambda'_1-\lambda'_2]_p!\cdots}.\end{aligned}\tag{10.5}$$

Substituting (10.4) and (10.5) to (10.3), we obtain

$$\begin{aligned}S_p(\lambda)&=\frac{[n]_p!}{[n-m]_p!}\frac{[d-n]_p!}{[d-n-m]_p!}\frac{[d-m]_p!}{[d]_p!}\frac{[m]_p!}{[1]_p^m}\frac{[n-m]_p!}{[n-\lambda'_1]_p!}p^{-\sum_j\lambda'_j(d-m-n+\lambda'_j)}\\&\quad\times\frac{[1]_p^m}{[m-\lambda_1]_p![\lambda'_1-\lambda'_2]_p!\cdots}\\&=\frac{\begin{bmatrix}d\\m+n\end{bmatrix}_p}{\begin{bmatrix}d\\m\end{bmatrix}_p\begin{bmatrix}d\\n\end{bmatrix}_p}\frac{[m+n]!}{[m-\lambda'_1]_p![n-\lambda'_1]_p!}\prod_{j\geq 1}\frac{1}{[\lambda'_j-\lambda'_{j+1}]_p!}p^{-\sum_{j\geq 1}\lambda'_j(d-m-n+\lambda'_j)}\\&=\overline{\tau}^d_{m,n}(\lambda).\end{aligned}$$

10.1.2 The Real Limit of the q-Selberg Measures

Next let us look at the η-limit. Remember that the η-limit ⓷ is obtained by $q\to 1$, $\lambda_j\to\infty$ in such a way $q^{\lambda_j}\to y_j$ for some $y_j\in[0,1]$. For the case where η is real, we replace the parameter α by $\alpha/2$, β by $\beta/2$ and γ by $\gamma/2$, respectively.

First of all we look at the limit $q\to 1$ of the higher rank β-function $C_q^{\alpha,\beta,\gamma}$. Note that this is "balanced". Namely, the sum of the parameters in numerator in (10.2) is equal to the sum in denominator. Remember that

$$\Gamma_q(\alpha)=\frac{\zeta_q(\alpha)}{\zeta_q(1)}(1-q)^{1-\alpha}\to\Gamma(\alpha)\qquad(q\to 1).$$

Hence we have

$$(1-q)^m C_q^{\alpha,\beta,\gamma}\to C_\eta^{\alpha,\beta,\gamma}:=\prod_{1\leq j\leq m}\frac{\zeta_\eta(\alpha+\gamma(j-1))\zeta_\eta(\beta+\gamma(m-j))\zeta_\eta(j\gamma)}{\zeta_\eta(\alpha+\beta+\gamma(m+j-2))\zeta_\eta(\gamma)}.$$

Note also that, when $q \to 1$, $\lambda \to \infty$ and $q^\lambda \to y$, the q-beta sum converges as follows;

$$\begin{aligned}\frac{\zeta_q(\alpha+\lambda)}{\zeta_q(\beta+\lambda)} &= \sum_{n\geq 0}\frac{\zeta_q(1)}{\zeta_q(1+n)}\frac{\zeta_q(\beta-\alpha+n)}{\zeta_q(\beta-\alpha)}q^{(\alpha+\lambda)n}\\ &= \sum_{n\geq 0}\frac{(1-q^{\beta-\alpha})(1-q^{\beta-\alpha+1})\cdot(1-q^{\beta-\alpha+n-1})}{(1-q)\cdots(1-q^n)}q^{\alpha n}q^{\lambda n}\\ &\to \sum_{n\geq 0}\frac{(\beta-\alpha)(\beta-\alpha+1)\cdots(\beta-\alpha+n-1)}{1\cdots n}y^n\\ &= (1-y)^{\alpha-\beta}.\end{aligned}$$

Therefore, since

$$\frac{\zeta_q(\alpha+\lambda_j+(m-j)\gamma)}{\zeta_q(1+\lambda_j+(m-j)\gamma)} \to (1-y_j)^{\alpha-1},$$

$$q^{\lambda_j(\beta+2\gamma(j-1))} \to y_j^{\beta+2\gamma(j-1)},$$

$$\frac{\zeta_q\big(\lambda_j-\lambda_i+\gamma(1+i-j)\big)}{\zeta_q(1+\lambda_j-\lambda_i+\gamma(i-j-1))}\big(1-q^{\lambda_j-\lambda_i+\gamma(i-j)}\big) \to \Big(1-\frac{y_j}{y_i}\Big)^{2\gamma-1}\Big(1-\frac{y_j}{y_i}\Big)$$

and also

$$(1-q)\delta_{q^{\lambda_j}} \to \frac{dy_j}{y_j},$$

we have

$$S_q(\lambda) \to S_\eta(\lambda) := \frac{1}{C_\eta^{2\cdot(\alpha,\beta,\gamma)}}\prod_{1\leq j\leq m}(1-y_j)^{\alpha-1}y_j^{\beta+2\gamma(j-1)}\prod_{i<j}\big(1-\frac{y_j}{y_i}\big)^{2\gamma}\frac{dy_1}{y_1}\cdots\frac{dy_m}{y_m}.$$

Replacing the parameters α by $\alpha/2$, β by $\beta/2$ and γ by $\gamma/2$, respectively, we obtain

$$\begin{aligned}S_\eta(\lambda) &= \frac{1}{C_\eta^{\alpha,\beta,\gamma}}\prod_{1\leq j\leq m}(1-y_j)^{\frac{\alpha}{2}-1}y_j^{\frac{\beta}{2}-1}\prod_{i<j}|y_i-y_j|^\gamma dy_1\cdots dy_m\\ &= \tau_\eta^{\alpha,\beta,\gamma}.\end{aligned}$$

This is the real Selberg measure which is calculated as the measure induced by Haar measure on the space of the relative positions of m-planes and n-planes.

10.2 Higher Rank q-Jacobi Basis

Remember that the q-Selberg measure $S_q(\lambda)$ is the probability measure on Λ_m. Here we would like to see it as a functional on the space of symmetric polynomials $\mathbb{C}[y_1,\ldots,y_m]^{\mathfrak{S}_m}$ by just putting the mass $S_q(\lambda)$ at the point

$y = (y_1, \ldots, y_m)$ where $y_j := q^{\lambda_j + \gamma(m-j)}$. We write $y = q^{\lambda+\gamma\rho}$ for this point. Then we get the inner product on $\mathbb{C}[y_1, \ldots, y_m]^{\mathfrak{S}_m}$ defined by

$$(f, g)_{q,m}^{\alpha,\beta,\gamma} := \sum_{\lambda} S_q(\lambda) f(q^{\lambda+\gamma\rho}) \overline{g(q^{\lambda+\gamma\rho})}$$

and the norm

$$||f||^2 = (f, f)_{q,m}^{\alpha,\beta,\gamma}.$$

The higher rank Little q-Jacobi polynomials $\varphi_{q(\lambda),m}^{\alpha,\beta,\gamma} \in \mathbb{C}[y_1, \ldots, y_m]^{\mathfrak{S}_m}$, which are basis for $\mathbb{C}[y_1, \ldots, y_m]^{\mathfrak{S}_m}$, are defined as follows;

(i) The leading term:

$$\varphi_{q(\lambda),m}^{\alpha,\beta,\gamma} = d_\lambda \cdot \mathfrak{m}_\lambda + (\text{lower terms}),$$

where $d_\lambda \neq 0$ and $\mathfrak{m}_\lambda$ is the monomial basis for $\mathbb{C}[y_1, \ldots, y_m]^{\mathfrak{S}_m}$.
(ii) Orthogonality:

$$(\varphi_{q(\lambda),m}^{\alpha,\beta,\gamma}, \mathfrak{m}_\mu)_{q,m}^{\alpha,\beta,\gamma} = 0$$

for all $\mu < \lambda$.
(iii) Normalization:

$$||\varphi_{q(\lambda),m}^{\alpha,\beta,\gamma}||^2 = \varphi_{q(\lambda),m}^{\alpha,\beta,\gamma}(0).$$

Note that in [Sto], Stokman normalize them to be monic, that is, $d_\lambda = 1$.

The shifted Macdonald polynomial $P_\lambda^*(y) = P_{q(\lambda)}^{*\gamma}(y_1, \ldots, y_m)$ is a polynomial of degree $|\lambda|$ and is symmetric in $y_j q^{-j\gamma}$. It holds that

$$P_\lambda^*(q^\mu) = 0 \qquad \text{unless } \lambda \subseteq \mu,$$
$$P_\lambda^*(q^\lambda) = (-1)^{|\lambda|} q^{\sum_j \lambda'_j (j-1) - 2\gamma\lambda_j(j-1)} \prod_{(i,j)\in\lambda} \left(1 - q^{\lambda_i - j + 1 + \gamma(\lambda'_j - i)}\right) \neq 0.$$

Note that $P_\lambda^*(q^\mu)$ is the eigenvalue of q-immanent, which is in the center of the enveloping algebra of the quantum group U_d (see below) acting on the representation of the highest weight μ. The generalized binomial coefficients are given by

$$\binom{\mu}{\lambda}_{q,\gamma} := \frac{P_\lambda^*(q^\mu)}{P_\lambda^*(q^\lambda)}.$$

This is essentially the cellular basis $C_{q(\lambda)}^\gamma$ for $\mathbb{C}[y_1, \ldots, y_m]^{\mathfrak{S}_m}$;

$$C_{q(\lambda)}^\gamma(y_1, \ldots, y_m) := \frac{1}{P_\lambda^*(q^\lambda)} P_\lambda^*(y_1 q^{\gamma(1-m)},\ y_2 q^{\gamma(2-m)}, \ldots, y_m)$$

Then taking the ⓟ or ⓡ limits, and setting $\gamma = 1$

$$C^{\gamma}_{q(\lambda)}(q^{\mu}) \to \begin{cases} \begin{pmatrix} \mu \\ \lambda \end{pmatrix}_p = \# X^{\mu}_{\lambda} & ⓟ, \\ \text{James and Constantine's } \begin{pmatrix} \mu \\ \lambda \end{pmatrix} & ⓡ \end{cases}$$

We here should comment about two natural ordering on partitions; One is the inclusion of the Young diagram, and the other is the dominant order. Namely, $\lambda \leq \mu$ if

$$|\lambda| = |\mu|, \qquad \sum_{i<j} \lambda_i \leq \sum_{i<j} \mu_i.$$

Now take a total ordering which refines both these partial ordering (e.g., the lexicographical ordering). Then the change of basis matrix between the monomial basis $\{\mathfrak{m}_\lambda\}$ and the cellular basis $\{C^{\gamma}_{q(\lambda)}\}$ is lower triangular, whence the idempotent basis $\{\varphi^{\gamma}_{q(\lambda)}\}$ is obtained by the Gram–Schmidt process applied to $\{C^{\gamma}_{q(\lambda)}\}$ (and the normalization $||\varphi^{\gamma}_{q(\lambda)}||^2 = \varphi^{\gamma}_{q(\lambda)}(0)$). Since the measure S_q converges to its p-adic or real counterparts, and so does the cellular basis, it follows that the idempotents $\varphi^{\gamma}_{q(\lambda)}$ also converge to their p-adic or real counterparts in the ⓟ and ⓡ limit, respectively. See [On1].

10.3 Quantum Groups

10.3.1 Higher Rank Quantum Groups

The quantum group $A_q(d) := \mathbb{C}_q[U_d]$ is a non-commutative deformation of the algebra of polynomial functions on U_d and is generated by t_{ij} for $1 \leq i, j \leq d$ and D_q^{-1} with relations

$$\begin{aligned} t_{ki}t_{kj} &= qt_{kj}t_{ki}, \quad t_{ik}t_{jk} = qt_{jk}t_{ik} \qquad (i < j), \\ t_{ii}t_{kj} &= t_{kj}t_{ii}, \\ t_{ij}t_{kl} - t_{kl}t_{ij} &= (q - q^{-1})t_{il}t_{kj} \qquad (i < k, j < l). \end{aligned} \tag{10.6}$$

We can write these relations by using the universal R-matrix

$$R = \sum_{1 \leq i,j \leq d} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji}.$$

as follows;

$$RT_1T_2 = T_2T_1R,$$

where $T_1 := (t_{ij}) \otimes I$ and $T_2 := I \otimes (t_{ij})$. If we define the quantum determinant D_q by

$$D_q := \sum_{\sigma \in \mathfrak{S}_d} (-q)^{l(\sigma)} t_{1\sigma(1)} \cdots t_{d\sigma(d)},$$

we see that D_q is a central element and we add the further relations $D_q \cdot D_q^{-1} = D_q^{-1} \cdot D_q = 1$. It is easy to see that $A_q(d)$ has the Hopf algebra structure via the following maps;

$$\begin{aligned} \Delta t_{ij} &:= \sum_k t_{ik} \otimes t_{kj}, \qquad \Delta(D_q) := D_q \otimes D_q, \\ \varepsilon(t_{ij}) &:= \delta_{ij}, \qquad \varepsilon(D_q) = 1, \\ S(t_{ij}) &:= (-q)^{i-j} D_{i,j} D_q^{-1}, \qquad S(D_q) := D_q^{-1}, \end{aligned}$$

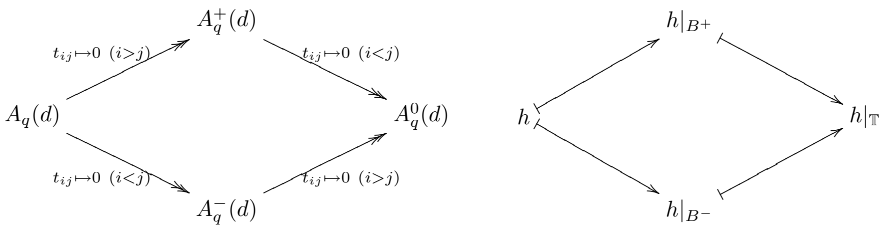
where

$$D_{i,j} := \sum_{\sigma \in \mathfrak{S}_{d-1}} (-q)^{l(\sigma)} t_{i_1, j_{\sigma(1)}} \cdots t_{i_{d-1} j_{\sigma(d-1)}}$$

and $\{i_1 < \cdots < i_{d-1}\} \cup \{j\} = \{1, \ldots, d\} = \{j_1 < \cdots < j_{d-1}\} \cup \{i\}$. $A_q(d)$ has also the $*$-structure

$$t_{ij}^* := S(t_{ji}).$$

We have the Hopf $*$-algebra homomorphisms defined as follows:



where

$$\begin{aligned} A_q^+(d) &= A_q(d)\big|_{B^+}, \\ A_q^-(d) &= A_q(d)\big|_{B^-}, \\ A_q^0(d) &= A_q(d)\big|_{\mathbb{T}} = \mathbb{C}[z_1^{\pm}, \ldots, z_d^{\pm}], \quad z_i := t_{ii}|_{\mathbb{T}}. \end{aligned}$$

Now we look at the right (resp. left) $A_q(d)$-comodules $\mathcal{L}$, that is, $\rho : \mathcal{L} \to \mathcal{L} \otimes A_q(d)$ (resp. $\rho : \mathcal{L} \to A_q(d) \otimes \mathcal{L}$). We say that $\mathcal{L}$ is right (resp. left) λ-highest weight if

$$(\mathrm{id}_{\mathcal{L}} \otimes |_{B^+}) \circ \rho(v) = v \otimes z^{\lambda}$$
$$\big(\text{resp. } (|_{B^-} \otimes \mathrm{id}_{\mathcal{L}}) \circ \rho(v) = z^{\lambda} \otimes v\big)$$

for any $v \in \mathcal{L}$, where $z^\lambda := t_{11}^{\lambda_1} \cdots t_{dd}^{\lambda_d}$. Such a right, say, $A_q(d)$-comodule is called unitary if we have a positive definite inner product $(\cdot,\cdot)$ on $\mathcal{L}$ which is invariant under the coaction, that is,

$$\sum_{i,j} (v_i^{(1)}, w_j^{(1)})(w_j^{(2)})^* v_i^{(2)} = (v,w) \cdot 1$$

for any $v, w \in \mathcal{L}$ with

$$\rho(v) = \sum_i v_i^{(1)} \otimes v_i^{(2)} \qquad (v_i^{(1)} \in \mathcal{L},\ v_i^{(2)} \in A_q(d)),$$
$$\rho(w) = \sum_i w_i^{(1)} \otimes w_i^{(2)} \qquad (w_i^{(1)} \in \mathcal{L},\ w_i^{(2)} \in A_q(d)).$$

Let $\overline{\rho}$ be the left $A_q(d)$-comodule structure on $\overline{\mathcal{L}}$ where $\overline{\mathcal{L}}$ is the complex conjugate of $\mathcal{L}$. Namely,

$$\overline{\rho} := (* \circ S) \otimes \mathrm{id}_{\mathcal{L}} \circ \sigma \circ \rho$$

where σ is the flip. Then the matrix coefficient $\theta : \overline{\mathcal{L}} \otimes \mathcal{L} \to A_q(d)$ is defined by

$$\theta(v,w) := \sum_i (w_i^{(1)}, v) w_i^{(2)}.$$

Notice that

$$\Delta \circ \theta = (\theta \otimes \mathrm{id}) \circ (\mathrm{id}_{\overline{\mathcal{L}}} \otimes \rho) = (\mathrm{id} \otimes \theta) \circ (\overline{\rho} \otimes \mathrm{id}_{\mathcal{L}}).$$

Moreover, it holds that

$$\theta(v,w) = (* \circ S)\theta(w,v).$$

There exists a unique irreducible λ-highest weight right comodule $V_R(\lambda)$ (the highest weight comodule is unique up to isomorphism). Let v_i be basis for $V_R(\lambda)$ and $\pi_i := (\rho(v_i), v_i) \in A_q(d)$. Then the matrix coefficient $\chi_\lambda := \sum_i \pi_i \in A_q(d)$ is a character of $V_R(\lambda)$ and $\chi_\lambda\big|_{\mathbb{T}} = s_\lambda(z)$, where $s_\lambda(z)$ is the Schur function (see [NYM]). Then $V_L(\lambda) := \mathrm{Hom}(V_R(\lambda), \mathbb{C})$ is an irreducible λ-highest weight left comodule.

Note that the comultiplication $\Delta : A_q(d) \to A_q(d) \otimes A_q(d)$ makes $A_q(d)$ into both left and right $A_q(d)$-comodule. Then the algebra $A_q(d)$ decomposed as follows:

$$A_q(d) = \bigoplus_{\lambda \in \Lambda_d} W(\lambda), \qquad W(\lambda) := V_L(\lambda) \otimes V_R(\lambda).$$

For any λ-highest weight right comodule $\mathcal{L}$, the matrix coefficient gives an isomorphism $\theta : \overline{\mathcal{L}} \otimes \mathcal{L} \xrightarrow{\sim} W(\lambda)$. The Haar measure $\tau : A_q(d) \to \mathbb{C}$ can be characterized as the unique invariant measure such that $\tau(W(\lambda)) = 0$ for $\lambda \neq 0$ and $\tau(1) = 1$. Moreover, if we set

$$\langle h_1, h_2 \rangle := \tau(h_2^* h_1),$$

this provides an inner product on $A_q(d)$.

10.3.2 The Universal Enveloping Algebra

We introduce the universal enveloping algebra $\mathrm{U}_q = \mathrm{U}_q(U_d)$ of $A_q(d)$, which is generated by the elements $L^{\pm}_{ij}$ where L^+ (resp. L^-) is upper (resp. lower) triangular and $L^{\pm}_{ii} = q^{\pm\tilde{\varepsilon}_i}$ with $\tilde{\varepsilon}_i \in \mathfrak{h} =: \mathrm{Lie}(\mathbb{T})$ is the dual of e_{ii} with relations

$$\begin{aligned} R^+ L^\delta_1 L^\delta_2 &= L^\delta_2 L^\delta_1 R^+, \\ R^+ L^+_1 L^-_2 &= L^-_2 L^+_1 R^+, \end{aligned}$$

where $\delta = \pm$, $L_1 = L \otimes 1$, $L_2 = 1 \otimes L$, $R^+ = PRP$ and $P = \sum_{i,j} e_{ij} \otimes e_{ji}$. Then U_q has the Hopf $*$-algebra structure via

$$\begin{aligned} \Delta(L^{\pm}_{ij}) &:= \sum_k L^{\pm}_{ik} \otimes L^{\pm}_{kj}, \\ \varepsilon(L^{\pm}_{ij}) &:= \delta_{ij}, \\ (L^{\pm}_{ij})^* &:= S(L^{\mp}_{ij}). \end{aligned}$$

Note that $* \circ S : L^{\pm}_{ij} \mapsto L^{\mp}_{ji}$ is an involution. We have the Hopf $*$-algebra duality

$$(\cdot, \cdot) : \mathrm{U}_q \otimes A_q(d) \to \mathbb{C}$$

given by

$$(L^{\pm}, T) = R^{\pm}, \qquad (L^{\pm}, D_q) = q^{\pm 1} \cdot 1.$$

This is a non-degenerate bilinear map. Moreover, it holds that

$$\begin{aligned} (u, a_1 a_2) &= (\Delta u, a_1 \otimes a_2), \qquad (u, 1) = \varepsilon(u), \\ (u_1 u_2, a) &= (u_1 \otimes u_2, \Delta a), \qquad (1, a) = \varepsilon(a), \\ (Su, a) &= (u, Sa), \\ (u, a^*) &= \overline{(* \circ Su, a)}, \qquad (u^*, a) = \overline{(u, * \circ Sa)}. \end{aligned}$$

It induce a $*$-algebra duality $\mathrm{U}_q(h) \otimes A^0_q(d) \to \mathbb{C}$ where $\mathrm{U}_q(h) := \mathbb{C}[q^{\pm\tilde{\varepsilon}_1}, \ldots, q^{\pm\tilde{\varepsilon}_d}]$, with $(q^h, z^\lambda) := q^{\langle h, \lambda \rangle}$.

Let $\rho : \mathcal{L} \to \mathcal{L} \otimes A_q(d)$ be a right $A_q(d)$-comodule. Then $\mathcal{L}$ becomes a left U_q-module as follows:

$$X \cdot v := \sum_i (X, v^{(2)}_i) v^{(1)}_i \qquad (v \in \mathcal{L}).$$

We call a vector $v \in \mathcal{L}$ a weight λ vector if $q^h \cdot v = q^{\langle h, \lambda \rangle} \cdot v$. Moreover, such a v is called highest weight if $L^+_{ij} \cdot v = 0$ for all $i < j$. $A_q(d)$ itself becomes a U_q-bimodule via

$$L^{\pm}_1 \cdot T_2 = T_2 \cdot R^{\pm}, \qquad T_2 \cdot L^{\pm}_1 = R^{\pm} \cdot T_2,$$

where $R^- := R^{-1}$. Let us write the decomposition of $A_q(d)$ as U_q-bimodule as

$$A_q(d) = \bigoplus_{\lambda \in \Lambda_d} W(\lambda).$$

Note that this is the eigenspace decomposition with respect to the center of U_q. In the center of U_q, we have the Casimir element (Laplace Beltrami operator) $\mathcal{C}$ defined by

$$\mathcal{C} := \sum_{i,j} q^{2(d-i)} L^+_{ij} S(L^-_{ji}).$$

This acts on $W(\lambda)$ as multiplication by the scalar $\chi_\lambda = \sum_{j \le d} q^{2(\lambda_j + d - j)}$.

10.3.3 Quantum Grassmann Manifolds

Let $\pi_m : A_q(d) \to A_q(d-m) \otimes A_q(m)$ be the homomorphism corresponding to the injection $B_m = \mathrm{U}(d-m) \times \mathrm{U}(m) \hookrightarrow \mathrm{U}(d)$. Then the quantum Grassmann manifolds $A_q(X)$ is defined by

$$A_q(X) := A_q(\mathrm{U}(d)/B_m) = \big\{ a \in A_q(d) \,\big|\, (\mathrm{id} \otimes \pi_m) \circ \Delta a = a \otimes 1 \big\}.$$

This is a $*$-subalgebra of $A_q(d)$ and a left $A_q(d)$-comodule. For a right $A_q(d)$-comodule $\rho : \mathcal{L} \to \mathcal{L} \otimes A_q(d)$, we denote by

$$\mathcal{L}^{B_m} := \big\{ v \in \mathcal{L} \,\big|\, (\mathrm{id}_m \otimes \pi_m) \circ \rho v = v \otimes 1 \big\}$$

the set of B_m-invariant vectors in $\mathcal{L}$. Let $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) be a right $A_q(d-m)$- (resp. $A_q(m)$-) comodule. Then $\mathcal{L}_1 \otimes \mathcal{L}_2$ becomes a right $A_q(d-m) \otimes A_q(m)$-comodule via $\rho = \sigma_{1,2} \circ \rho_1 \otimes \rho_2$ where $\sigma_{1,2}$ is the flip. Let $\mathcal{L}$ be a right $A_q(d)$ comodule. Then it is also a right $A_q(B_m)$-comodule via $(\mathrm{id} \otimes \pi_m) \circ \rho$. Then, as a $A_q(B_m)$-module, $V_R(\lambda)$ decomposed as follows:

$$V_R(\lambda) = \bigoplus_{\lambda_1, \lambda_2} \big(V_R(\lambda_1) \otimes V_R(\lambda_2) \big)^{\oplus c^\lambda_{\lambda_1, \lambda_2}}.$$

Here $c^\lambda_{\lambda_1,\lambda_2} \in \mathbb{N}$ is the Littlewood–Richardson coefficient, that is,

$$s_\lambda(z_1, \ldots, z_d) = \sum_{\lambda_1, \lambda_2} c^\lambda_{\lambda_1,\lambda_2} s_{\lambda_1}(z_1, \ldots, z_{d-m}) s_{\lambda_2}(z_{d-m+1}, \ldots, z_d).$$

Theorem 10.3.1 ([DS]). *We have*

$$A_q(X)^{B_m} = A_q(B_m \backslash \mathrm{U}(d)/B_m) = \bigoplus_{\lambda \in \Lambda_m} \mathbb{C} \cdot \varphi_{q(\lambda)},$$

where $\varphi_{q(\lambda)}$ is the higher rank Little q-Jacobi polynomial with respect to the Selberg measure with parameters $\alpha = 1 + m - m = 1$, $\beta = 1 + d - 2m$ and $\gamma = 1$.

Let

$$J^\sigma := \sum_{1\le k\le m}(1-q^\sigma)e_{k,k} + \sum_{m\le k\le d-m} e_{k,k} - \sum_{1\le k\le m} q^\sigma(e_{k,d-k+1}+e_{d-k+1,k})$$

$$= \left(\begin{array}{ccc|ccc|ccc} 1-q^\sigma & & & & & & & & -q^\sigma \\ & \ddots & & & & & & \iddots & \\ & & 1-q^\sigma & & & & -q^\sigma & & \\ \hline & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ \hline & & -q^\sigma & & & & 0 & & \\ & \iddots & & & & & & \ddots & \\ -q^\sigma & & & & & & & & 0 \end{array}\right)$$

Then J^σ is a solution of the reflection equation

$$R_{12}J_1R_{12}^{-1}J_2 = J_2R_{21}^{-1}J_1R_{21}, \tag{10.7}$$

where $R_{12} = R$, $R_{21} = R^+$, $J_1 = J\otimes I$ and $J_2 = I\otimes J$. Let $\mathfrak{B}^\sigma \subseteq \mathrm{U}_q(d)$ be the subspace of $\mathrm{U}_q(d)$ spanned by the coefficients of $L^+J^\sigma - J^\sigma L^- \in \mathrm{End}(\mathbb{C}^{\oplus d})\otimes \mathrm{U}_q(d) = \mathrm{Mat}_{d,d}(\mathrm{U}_q(d))$. Then $\Delta\mathfrak{B}^\sigma \subseteq \mathrm{U}_q(d)\otimes\mathfrak{B}^\sigma + \mathfrak{B}^\sigma\otimes \mathrm{U}_q(d)$ is a coideal, $\varepsilon(\mathfrak{B}^\sigma) = 0$ and $*\circ S(\mathfrak{B}^\sigma) = \mathfrak{B}^\sigma$. Let

$$A_q({}^\sigma X) := \{a\in A_q(d) \mid \mathfrak{B}^\sigma\cdot a = 0\},$$
$$A_q(X^\sigma) := \{a\in A_q(d) \mid a\cdot \mathfrak{B}^\sigma = 0\}.$$

Then $A_q({}^\sigma X)$ (resp. $A_q(X^\sigma)$) is $*$-subalgebra, a right (resp. left) $\mathrm{U}_q(d)$-module and left (resp. right) $A_q(d)$-subcomodule. Then we have $\lim_{\sigma\to\infty} A_q({}^\sigma X) = A_q(X)$. Put $\mathfrak{B}^\infty := \lim_{\sigma\to\infty}\mathfrak{B}^\sigma =$ "$\mathrm{U}_q(d-m)\otimes\mathrm{U}_q(m)$", being invariant with respect to $\mathfrak{B}^\infty$ is equivalent to being co-invariant with respect to π_m. Note that $\mathfrak{B}^\infty$ strictly contains the subspace generated by the coefficients of $L^+J^\infty - J^\infty L^-$ where $J^\infty := \lim_{\sigma\to\infty} J^\sigma = \begin{pmatrix} I_{d-m} & \\ & 0_m\end{pmatrix}$. Then $\mathfrak{B}^\infty$ is co-invariant with respect to π_m: namely, for any $v\in\mathfrak{B}^\infty$, $(\mathrm{id}\otimes\pi_m)\circ\rho v = v\otimes 1$. It is shown that J^σ satisfies the reflection equation (10.7) if and only if $W^\sigma := \sum_{i,j} J^\sigma_{i,j} e_i\otimes e_j^* \in \mathrm{End}(\mathbb{C}^{\oplus d}) = V\otimes V^*$ is a $\mathfrak{B}^\sigma$-fixed vector. Then we have

Theorem 10.3.2 ([NDS]). *The left (resp. right)* $\mathrm{U}_q(d)$*-module* $V_R(\lambda)$ *(resp.* $V_L(\lambda)$*) has at most* 1 B^σ*-fixed vector. It exists if and only if*

$$\lambda\in\Lambda_m \simeq \{(\lambda_1,\dots,\lambda_m,0,\dots,0,-\lambda_m,\dots,-\lambda_1)\} \subseteq P^+ = \{(\lambda_1,\dots,\lambda_d)\}.$$

Let

$$\mathcal{H}^{\sigma,\tau} = A_q(\mathfrak{B}^\sigma\backslash \mathrm{U}_q(d)/\mathfrak{B}^\tau) := \{a\in A_q(d) \mid \mathfrak{B}^\sigma\cdot a = a\cdot\mathfrak{B}^\tau = 0\}.$$

Then $\mathcal{H}^{\sigma,\tau}$ is $*$-subalgebra of $A_q(d)$. It decomposes as follows:

$$\mathcal{H}^{\sigma,\tau} = \bigoplus_{\lambda\in\Lambda_m} \mathcal{H}^{\sigma,\tau}(\lambda), \qquad \mathcal{H}^{\sigma,\tau}(\lambda) := \mathcal{H}^{\sigma,\tau} \cap W(\lambda).$$

It can be shown that $\dim\mathcal{H}^{\sigma,\tau}(\lambda) = 1$. Hence there exist $\varphi_\lambda^{\sigma,\tau} \in \mathcal{H}^{\sigma,\tau}(\lambda)$ such that $\mathcal{H}^{\sigma,\tau}(\lambda) = \mathbb{C}\cdot\varphi_\lambda^{\sigma,\tau}$. The functions $\varphi_\lambda^{\sigma,\tau}$ are identified in [NDS] with the Koornwinder polynomials which are a higher rank analogue of the Askey–Wilson polynomials. Because the Casimir element $C \in \mathrm{U}_q(d)$ is a central element, we have $C : \mathcal{H}^{\sigma,\tau} \to \mathcal{H}^{\sigma,\tau}$, whence $C_{\sigma,\tau} := C\big|_{\mathcal{H}^{\sigma,\tau}}$: $\mathcal{H}^{\sigma,\tau}\big|_{\mathbb{T}} \to \mathcal{H}^{\sigma,\tau}\big|_{\mathbb{T}}$. Note that $\mathcal{H}^{\sigma,\tau}\big|_{\mathbb{T}} \subseteq \mathbb{C}[z_1^{\pm},\dots,z_m^{\pm}]$. The operator $C_{\sigma,\tau} - \chi_\lambda$ is identified with the Koornwinder operator. Put $x_j := z_j\cdot z_{d-j+1}^{-1}$. Then we have $x_j\cdot\mathrm{diag}(a_{11},\dots,a_{mm},1,\dots,1,a_{mm}^{-1},\dots a_{11}^{-1}) = a_{jj}^2$ since $z_j \cdot \mathrm{diag}(a_{11},\dots,a_{mm}) = a_{jj}$. Put $W := \mathfrak{S}_m \ltimes (-1)^m$. Then we have

$$\mathcal{H}^{\sigma,\tau} = \mathbb{C}[x_1^{\pm},\dots,x_m^{\pm}]^W = \mathbb{C}[e_1^{\sigma,\tau},\dots,e_m^{\sigma,\tau}],$$

via $P \mapsto \hat{P}$ where $P(x_1,\dots,x_m) = \hat{P}(e_1^{\sigma,\tau}(y),\dots,e_m^{\sigma,\tau}(y))$ and $y_j := \frac{1}{2}(1-\frac{x_j+x_j^{-1}}{2})$. Here $e_j^{\sigma,\tau} = \mathfrak{m}_{1^j} \in \mathcal{H}^{\sigma,\tau}$ is the elementary symmetric function. Let $\tau : \mathcal{H}^{\sigma,\tau} \to \mathbb{C}$ be the Haar measure. Put $\tau(\hat{P}(e_1^{\sigma,\tau},\dots,e_m^{\sigma,\tau})) = \frac{\langle P\rangle_{\sigma,\tau}}{\langle 1\rangle_{\sigma,\tau}}$, which is identified with the normalized Koornwinder weight. In the limit $\sigma\to\infty$ (resp. $\sigma,\tau\to\infty$), we have (cf. [DS])

$$\begin{aligned} \varphi_\lambda^{\infty,\tau} &= \hat{P}_\lambda^B(\widetilde{e}_1^{\infty,\tau},\dots,\widetilde{e}_m^{\infty,\tau};1,q^{2(d-2m)},1,q^{2\tau+2(d-2m)};q^2,q^2) \\ \big(\text{resp. } \varphi_\lambda^{\infty,\infty} &= \hat{P}_\lambda^L(\widetilde{e}_1^{\infty,\infty},\dots,\widetilde{e}_m^{\infty,\infty};1,q^{2(d-2m)},1;q^2,q^2)\big), \end{aligned}$$

where P_λ^B (resp. P_λ^L) is the multi-variable Big (resp. Little) q-Jacobi polynomial, and where

$$\widetilde{e}_r^{\infty,\tau} := (-1)^r \lim_{\sigma\to\infty} q^{r(\sigma+\tau-1)} e_r^{\sigma,\tau}, \qquad \widetilde{e}_r^{\infty,\infty} := (-1)^r \lim_{\sigma\to\infty} q^{r(2\sigma-1)} e_r^{\sigma,\sigma}.$$

11

Quantum Group $\mathrm{U}_q(\mathfrak{su}(1,1))$ and the q-Hahn Basis

Summary. In Chap. 11 we introduced the Hopf algebra U_q and discussed the p-adic limit and the real limit of the subalgebras U_q^+ and U_q^-. These algebras are interchanged by the antipode; $S : \mathrm{U}_q^+ \xrightarrow{\sim} \mathrm{U}_q^-$. We defined the β-highest U_q-module V^β and show its uniqueness. Comparing the norm of the basis v_n^β with the q-Laguerre basis $\varphi_{q,n}^\beta$, we have a realization of V^β. Namely, we have the isomorphism

$$V^\beta = \bigoplus_{n\geq 0} \mathbb{C}v_n^\beta \longrightarrow H_{\mathbb{Z}_q}^\beta = \bigoplus_{n\geq 0} \mathbb{C}\varphi_{q,n}^\beta; \qquad v_n^\beta \longmapsto \varphi_{q,n}^\beta.$$

We show the Clebsch–Gordan matrix which interchanges the natural weight basis for $V^\alpha \otimes V^\beta$ is given in terms of the q-Hahn basis $\varphi_{q,m}^{(\alpha)\beta}(i,j)$. We then show how the universal R-matrix repairs the lost symmetry in the parameter α, β of the non-symmetric q-β-chain.

11.1 The Quantum Universal Enveloping Algebra $\mathrm{U}_q(\mathfrak{su}(1,1))$

11.1.1 Deformation of $\mathrm{U}(\mathfrak{sl}(2,\mathbb{C}))$

The *Quantum group* $\mathrm{U}_q = \mathrm{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}(2,\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. The quantum group $\hat{\mathrm{U}}_t = \hat{\mathrm{U}}_t(\mathfrak{sl}(2,\mathbb{C}))$ is the complete $\mathbb{C}[[t]]$-algebra generated by X_+, X_- and H with the following relations;

$$[H, X_\pm] = \pm 2X_\pm, \qquad [X_+, X_-] = \frac{\sinh(\frac{t}{2}H)}{\sinh(\frac{t}{2})} = \frac{e^{\frac{t}{2}H} - e^{-\frac{t}{2}H}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}.$$

When $t = 0$, these are the same as the commutation relations for the generators of $\mathrm{U}(\mathfrak{sl}(2,\mathbb{C}))$. We give the $*$-*structure* on $\hat{\mathrm{U}}_t$ as follows;

$$H^* = H, \quad X_\pm^* = -X_\mp.$$

The algebra $\hat{\mathrm{U}}_t$ with this $*$-structure is referred to $\hat{\mathrm{U}}_t(\mathfrak{su}(1,1))$. Remark that this $*$-structure is for $SU(1,1)$ but for $SL(2,\mathbb{R})$ (note that $SU(1,1)$ and $SL(2,\mathbb{R})$ are isomorphic as real Lie groups) we have another one. To work in the algebraic version, let us specialize formal parameters to numbers: we usually set $q := e^t \in (0,1)$. Consider the subalgebra $\mathrm{U}_{\langle q\rangle}$ of $\hat{\mathrm{U}}_t$ generated by $X_\pm$ and $q^{\pm\frac{H}{4}}$. In this case, the relations for these generators are given as follows;

$$\begin{aligned} q^{\frac{H}{4}} \cdot q^{-\frac{H}{4}} &= 1 = q^{-\frac{H}{4}} \cdot q^{\frac{H}{4}}, \\ q^{\frac{H}{4}} X_\pm q^{-\frac{H}{4}} &= q^{\pm\frac{1}{2}} X_\pm, \\ [X_+, X_-] &= \frac{q^{\frac{H}{2}} - q^{-\frac{H}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \langle H \rangle_{q^{\frac{1}{2}}} \end{aligned}$$

where

$$\langle H \rangle_q := \frac{q^H - q^{-H}}{q - q^{-1}}.$$

Note that $\langle H \rangle_q$ is symmetric in the following sense; $\langle H \rangle_{q^{-1}} = \langle H \rangle_q$. Further we have the subalgebra U_q of $\mathrm{U}_{\langle q\rangle}$, which is generated by $Y_+ := X_+ q^{\frac{H-1}{4}}$, $Y_- := -q^{\frac{H-1}{4}} X_-$ and $q^{\pm\frac{H}{2}}$. The relations for U_q are almost the same as the one before;

$$\begin{aligned} q^{\frac{H}{2}} \cdot q^{-\frac{H}{2}} &= 1 = q^{-\frac{H}{2}} \cdot q^{\frac{H}{2}}, \\ q^{\frac{H}{2}} Y_\pm q^{-\frac{H}{2}} &= q^{\pm} Y_\pm, \\ Y_- Y_+ - q Y_+ Y_- &= \frac{1 - q^H}{1 - q} = [H]_q. \end{aligned}$$

Actually, the last equation can be shown as follows:

$$\begin{aligned} Y_- Y_+ - q Y_+ Y_- &= -q^{\frac{H-1}{4}} X_- X_+ q^{\frac{H-1}{4}} + q X_+ q^{\frac{H-1}{4}} q^{\frac{H-1}{4}} X_- \\ &= q^{\frac{H-1}{4}} \Big(\frac{q^{\frac{H}{2}} - q^{-\frac{H}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \Big) q^{\frac{H-1}{4}} = \frac{1 - q^H}{1 - q}. \end{aligned}$$

However $[H]_q$ is non-symmetric, this formulation has much advantage of converging in the p-adic limit. The $*$-structure on U_q induced from the one on $\hat{\mathrm{U}}_t$ is given by

$$\big(q^{\pm\frac{H}{2}}\big)^* = q^{\pm\frac{H}{2}}, \quad (Y_\pm)^* = Y_\mp.$$

This shows that $q^{\pm\frac{H}{2}}$ is self-adjoint if q is real. The arithmetics of U_q are completely controlled by the following relation, which is a generalization of the above relation;

$$q^{n\frac{H}{2}} Y_\pm^m = q^{\pm nm} Y_\pm^m q^{n\frac{H}{2}} \qquad (m, n \in \mathbb{N}).$$

Further, by induction on n and m, we have

$$Y_-^n Y_+^m = \sum_{0\le k\le\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q [k]_q! q^{(n-k)(m-k)} \\ \times Y_+^{m-k}[H+m+n-k-1]_q \cdots [H+m+n-2k]_q Y_-^{n-k}.$$

11.1.2 The β-Highest Weight Representation

Let V be a Hilbert space and a $*$-representation $\mathrm{U}_q \to \mathrm{End}(V)$ (i.e., V is a U_q-module). We call V the *β-highest weight representation* if there exists a vector $v_0^\beta \in V$ satisfying the following conditions;

$$(i)\ Y_- v_0^\beta = 0,$$
$$(ii)\ q^{\frac{H}{2}} v_0^\beta = q^{\frac{\beta}{2}} v_0^\beta,$$
$$(iii)\ U_q \cdot v_0^\beta \text{ is dense in } V.$$

Then v_0^β is called a *β-highest weight vector* of U_q. We normalize it as $||v_0^\beta||_V = 1$. Let us study such a representation. Set

$$v_n^\beta := (-1)^n \frac{q^{-\frac{n\beta}{2}}}{[n]_q!} Y_+^n v_0^\beta \qquad (n \in \mathbb{N}).$$

Then we can easily obtain the following relations;

$$\begin{cases} q^{\frac{H}{2}} v_n^\beta = q^{\frac{\beta}{2}+n} v_n^\beta, \\ -Y_+ v_n^\beta = [n+1]_q q^{\frac{\beta}{2}} v_{n+1}^\beta, \\ -Y_- v_n^\beta = [n-1+\beta]_q q^{-\frac{\beta}{2}} v_{n-1}^\beta. \end{cases} \tag{11.1}$$

Note that the third equality can be obtained by induction on n. From (11.1), we see that v_n^β are orthogonal since they are eigenvectors with respect to the self-adjoint operator $q^{\frac{H}{2}}$ with different eigenvalues. Further, we see that $\mathrm{Span}_{\mathbb{C}}\{v_n\}_{n\ge 0}$ is closed with respect to the U_q-action. Hence we have the decomposition

$$V = \bigoplus_{n\ge 0} \mathbb{C} v_n^\beta. \tag{11.2}$$

The norms of v_n^β can be calculated as follows;

$$||v_n^\beta||_V^2 = -\frac{q^{-\frac{\beta}{2}}}{[n]_q}(Y_+ v_{n-1}^\beta, v_n^\beta) = -\frac{q^{-\frac{\beta}{2}}}{[n]_q}(v_{n-1}^\beta, Y_- v_n^\beta) \\ = \frac{q^{-\frac{\beta}{2}}}{[n]_q}[n-1+\beta]_q q^{-\frac{\beta}{2}}(v_{n-1}^\beta, v_{n-1}^\beta)$$

$$= q^{-\beta}\frac{1-q^{n-1+\beta}}{1-q^n}||v_{n-1}^\beta||_V^2.$$

Hence we have by induction

$$\begin{aligned}||v_n^\beta||_V^2 &= q^{-\beta n}\frac{(1-q^{n-1+\beta})\cdots(1-q^\beta)}{(1-q^n)\cdots(1-q)}||v_0^\beta||_V^2\\ &= q^{-\beta n}\frac{\zeta_q(1)}{\zeta_q(1+n)}\frac{\zeta_q(\beta+n)}{\zeta_q(\beta)} = C_q^\beta(n)\end{aligned}$$

since $||v_0^\beta||_V = 1$. Conversely, for any $\beta > 0$, define the Hilbert space V^β as the right hand side of (11.2) with the norms $||v_n^\beta||_{V^\beta}^2 = C_q^\beta(n)$ and the relations (11.1). Then V^β is isomorphic to the $*$-representation V. This shows that there exists a unique β-highest weight representation of U_q up to an isomorphism.

Comparing with the norm of the q-Laguerre basis $\varphi_{\mathbb{Z}_q,n}^\beta$, we have the isomorphism

$$V^\beta \xrightarrow{\sim} H_{\mathbb{Z}_q}^\beta; \qquad v_n^\beta \longmapsto \varphi_{\mathbb{Z}_q,n}^\beta.$$

Therefore the space $H_{\mathbb{Z}_q}^\beta$, which is the boundary space for the q-γ-process, is a realization of the β-highest weight representation of $SU(1,1)$. Then the U_q-actions (11.1) for the three operators Y_+, Y_- and $q^{\frac{H}{2}}$ are translated to the following action on $H_{\mathbb{Z}_q}^\beta$;

$$\begin{cases} q^{-\frac{H}{2}}\varphi(g^i) = q^{-i}\big[(q^{-\frac{\beta}{2}}+q^{\frac{\beta}{2}-1})\varphi(g^i) - (1-q^i)q^{-\frac{\beta}{2}}\varphi(g^{i-1}) - q^{\frac{\beta}{2}-1}\varphi(g^{i+1})\big], \\ Y_+q^{-\frac{H}{2}}\varphi(g^i) = \dfrac{q^{-i}}{1-q}\big[\varphi(g^i) - (1+q^{1-\beta})(1-q^i)\varphi(g^{i-1}) + q^{1-\beta}(1-q^i)(1-q^{i-1})\varphi(g^{i-2})\big], \\ q^{-\frac{H}{2}}Y_-\varphi(g^i) = \dfrac{q^{-i}}{1-q}\big[\varphi(g^i) - (1+q^{1-\beta})(1-q^i)\varphi(g^{i+1}) + q^{\beta-1}\varphi(g^{i+2})\big]. \end{cases}$$

From these we see that the action of $q^{-\frac{H}{2}}$, $Y_+q^{-\frac{H}{2}}$ and $q^{-\frac{H}{2}}Y_-$ are via second order difference operator. Let us denote by U_q^- the subalgebra of U_q generated by $Y_+q^{-\frac{H}{2}}$, $q^{-\frac{H}{2}}Y_-$ and $q^{-\frac{H}{2}}$ (without $q^{\frac{H}{2}}$). Then U_q^- acts on the space $H_{\mathbb{Z}_q}^\beta$ via finite difference operator. Note that the action of $q^{\frac{H}{2}}$ on $H_{\mathbb{Z}_q}^\beta$ is not via a difference operator.

Now we can understand the creation and the annihilation operators. Consider a β-highest representation V^β of U_q. Define the difference operator $D : V^\beta \to V^{\beta+1}$ as follows;

$$-Dv_n^\beta := \frac{q^{-\beta}}{1-q}v_{n-1}^{\beta+1}.$$

Put $D_\beta^+ := (1-q^\beta)D^*$ where D^* is the adjoint operator of D. Then we have

$$-D_\beta^+ v_{n-1}^{\beta+1} = [n]_q q^{1-n} v_n^\beta.$$

We again obtain the following ladder

$$V^\beta \underset{D_\beta^+}{\overset{D}{\rightleftarrows}} V^{\beta+1} \underset{D_{\beta+1}^+}{\overset{D}{\rightleftarrows}} V^{\beta+2}$$

and the Heisenberg relation up the ladder

$$DD_\beta^+ - D_{\beta+1}^+ D = \frac{q^{-\beta}}{1-q}\mathrm{id}_{V^{\beta+1}}.$$

One can obtain the following formula from the Heisenberg relation;

$$v_n^\beta = (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{[n]_q!}(D^+)^n v_n^{\beta+n}.$$

This gives an explanation of the Laguerre basis. Next we treat the real and p-adic limits.

11.1.3 Limits of the Subalgebras $\mathbf{U}_q^\pm$

Taking the real limit ⓡ (i.e., the limit $q \to 1$), we have $[H]_q \to H$. Hence we obtain the usual $\mathfrak{sl}(2,\mathbb{C})$-actions on the space $H_{\mathbb{Z}_\eta}^\beta = L^2\big(\mathbb{R}/\{\pm 1\}, \tau_{\mathbb{Z}_\eta}^\beta\big)$ given by

$$\begin{cases} H = x\dfrac{\partial}{\partial x} + \dfrac{\beta}{2} - \dfrac{1}{4\pi}\Big(\dfrac{\partial^2}{\partial x^2} + \dfrac{\beta-1}{x}\dfrac{\partial}{\partial x}\Big), \\ Y_- = \dfrac{1}{4\pi}\Big(\dfrac{\partial^2}{\partial x^2} + \dfrac{\beta-1}{x}\dfrac{\partial}{\partial x}\Big), \\ Y_+ = e^{\pi x^2}\dfrac{1}{4\pi}\Big(\dfrac{\partial^2}{\partial x^2} + \dfrac{\beta-1}{x}\dfrac{\partial}{\partial x}\Big)e^{-\pi x^2}. \end{cases}$$

Now we want to similarly take the p-adic limit. Let U_q^+ be the subalgebra of U_q generated by $Y_\pm$ and $q^{\frac{H}{2}}$ (without $q^{-\frac{H}{2}}$). The relations between these generator are given by

$$\begin{aligned} q^{\frac{H}{2}}Y_+ &= qY_+q^{\frac{H}{2}}, \\ q\cdot q^{\frac{H}{2}}Y_- &= Y_-q^{\frac{H}{2}}, \\ Y_-Y_+ - qY_+Y_- &= \frac{1-q^H}{1-q} = [H]_q. \end{aligned}$$

Consider now the p-adic limit ⓟ. In the p-adic limit, $q \to 0$ but $q^H \longrightarrow p^{-H}$. The algebra U_q^+ also converges to the algebra U_p^+, which is generated by Y_+, Y_- and $p^{-\frac{H}{2}}$ with relations

$$\begin{aligned} p^{-\frac{H}{2}}Y_+ &= 0, \\ 0 &= Y_-p^{-\frac{H}{2}}, \\ Y_-Y_+ &= 1 - p^{-H}. \end{aligned}$$

These act on the space

$$V_p^\beta = \bigoplus_{n\geq 0} \mathbb{C}v_n^\beta,$$

with the norm

$$||v_n^\beta||_{V_p^\beta} = \begin{cases} 1 & \text{if } n=0, \\ (1-p^{-\beta})p^{\beta n} & \text{if } n\geq 1. \end{cases}$$

The action of U_q^+ is explicitly given as follows;

$$\begin{cases} p^{-\frac{H}{2}}v_n^\beta = \begin{cases} p^{-\frac{\beta}{2}}v_0^\beta & \text{if } n=0, \\ 0 & \text{if } n\geq 1, \end{cases}, \\ -Y_+v_n^\beta = p^{-\frac{\beta}{2}}v_{n+1}^\beta, \\ -Y_-v_n^\beta = \begin{cases} 0 & \text{if } n=0, \\ (1-p^{-\beta})p^{\frac{\beta}{2}}v_0^\beta & \text{if } n=1,. \\ p^{\frac{\beta}{2}}v_{n-1}^\beta & \text{if } n\geq 2. \end{cases} \end{cases} \tag{11.3}$$

We obtained the isomorphism between the space V^β and the boundary space $H_{\mathbb{Z}_p}^\beta$ for the γ-measure as before;

$$V_p^\beta \longrightarrow H_{\mathbb{Z}_p}^\beta, \qquad v_n^\beta \longmapsto \varphi_{\mathbb{Z}_p,n}^\beta$$

Then the actions (11.3) of U_q^+ on the space $H_{\mathbb{Z}_p}^\beta$ is translated as the following formulas;

$$\begin{cases} p^{-\frac{H}{2}}\varphi(p^i) = p^{-\frac{\beta}{2}}(\varphi,\phi_{\mathbb{Z}_p})\mathbf{1}, \\ \quad Y_+\varphi(p^i) = p^{-\frac{\beta}{2}}(\varphi,\phi_{\mathbb{Z}_p})\mathbf{1} - p^{\frac{\beta}{2}}\varphi(p^{i-1}), \\ \quad Y_-\varphi(p^i) = p^{-\frac{\beta}{2}}(\varphi,\phi_{\mathbb{Z}_p})\mathbf{1} - p^{-\frac{\beta}{2}}\varphi(p^{i+1}). \end{cases}$$

From this we see that $p^{-\frac{H}{2}}$ is just the orthogonal projection on the vacuum $\mathbb{C}v_0^\beta$ (times $p^{-\frac{\beta}{2}}$), $Y_+ - p^{-\frac{H}{2}}$ is isometry of this space of rank 1 and $Y_- - p^{-\frac{H}{2}}$ which is the adjoint co-isometry. Therefore the algebra generated by $Y_\pm - p^{-\frac{H}{2}}$ is isomorphic to the Toeplitz algebra.

11.1.4 The Hopf Algebra Structure

In this subsection we study the following algebraic structures of the quantum group $\hat{\mathrm{U}}_t$ and its subalgebras $\mathrm{U}_{\langle q\rangle}$, U_q and $\mathrm{U}_q^\pm$;

$$\underbrace{\mathrm{U}_q^\pm}_{\text{bialgebras}} \subseteq \underbrace{\mathrm{U}_q \subseteq \mathrm{U}_{\langle q\rangle} \subseteq \hat{\mathrm{U}}_t}_{\text{Hopf algebras}}$$

First we see that $\hat{U}_t$ is a *bialgebras*. The *comultiplication* $\Delta : \hat{\mathrm{U}}_t \to \hat{\mathrm{U}}_t \hat{\otimes} \hat{\mathrm{U}}_t$ (here $\hat{\otimes}$ is the completed tensor product) on $\hat{\mathrm{U}}_t$ is defined by

$$\Delta(H) = 1 \otimes H + H \oplus 1, \qquad \Delta(X_\pm) = q^{-\frac{H}{4}} \otimes X_\pm + X_\pm \otimes q^{\frac{H}{4}}.$$

It induces the comultiplications $\Delta : \mathrm{U}_q \to \mathrm{U}_q \hat{\otimes} \mathrm{U}_q$ and $\Delta : \mathrm{U}_q^\pm \to \mathrm{U}_q^\pm \hat{\otimes} \mathrm{U}_q^\pm$ as follows;

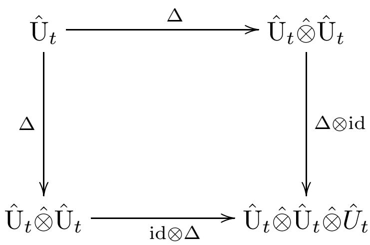
$$\Delta(q^{\pm\frac{H}{2}}) = q^{\pm\frac{H}{2}} \otimes q^{\pm\frac{H}{2}}, \qquad \Delta(Y_\pm) = 1 \otimes Y_\pm + Y_\pm \otimes q^{\frac{H}{2}}.$$

The map Δ is a $*$-*homomorphism* in the sense that

$$\Delta(h_1 h_2) = \Delta(h_1)\Delta(h_2), \qquad \Delta \circ * = (* \otimes *) \circ \Delta$$

and satisfies the *coassociativity*

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta.$$



Let us write $\Delta(h) = \sum_i h_i^{(1)} \otimes h_i^{(2)}$. Then this equation can be written as

$$\sum_i h_i^{(1)} \otimes \Delta(h_i^{(2)}) = \sum_i \Delta(h_i^{(1)}) \otimes h_i^{(2)}.$$

The *counit* $\varepsilon : \hat{U}_t \to \mathbb{C}$ is gives by

$$\varepsilon(H) = 0, \qquad \varepsilon(X_\pm) = 0.$$

It also induces $\varepsilon : \mathrm{U}_q \to \mathbb{C}$ and $\varepsilon : \mathrm{U}_q^\pm \to \mathbb{C}$ as follows;

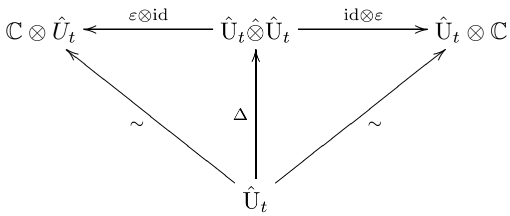
$$\varepsilon(q^{\pm\frac{H}{2}}) = 1, \qquad \varepsilon(Y_\pm) = 0.$$

The map ε is clearly $*$-homomorphism;

$$\varepsilon(h_1 h_2) = \varepsilon(h_1)\varepsilon(h_1), \qquad \varepsilon \circ * = * \circ \varepsilon,$$

where the map $*$ on $\mathbb{C}$ is the complex conjugation. It satisfies

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta.$$



Namely, using the notation above, we have

$$\sum_i \varepsilon(h_i^{(1)}) \cdot h_i^{(2)} = h = \sum_i h_i^{(1)} \varepsilon(h_i^{(2)}).$$

Moreover, $\hat{U}_t$ has an Hopf algebra structure; we have the *antipode* $S : \hat{U}_t \to \hat{U}_t$ defined by

$$S(H) = -H, \qquad S(X_\pm) = -q^{\pm\frac{1}{2}} X_\pm.$$

It also induces the antipode $S : U_q \to U_q$ as follows;

$$S(q^{\pm\frac{H}{2}}) = q^{\mp\frac{H}{2}}, \qquad S(Y_\pm) = -Y_\pm q^{\frac{H}{2}}.$$

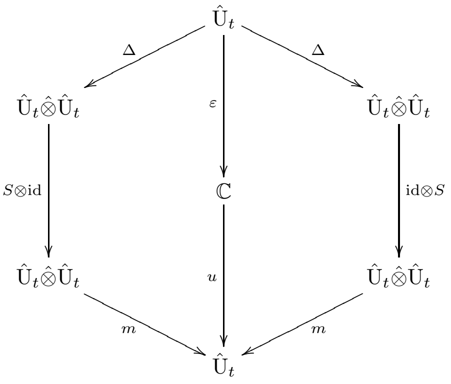
Note that it does not induce a map $U_q^\pm \to U_q^\pm$, however, it gives an isomorphism

$$S : U_q^+ \xrightarrow{\simeq} U_q^-.$$

These algebras $U_q^\pm$ are bialgebras but not Hopf algebras.

Satisfies the antipode axion

$$m\circ(S\otimes\mathrm{id})\circ\Delta = u\circ\varepsilon = m\circ(\mathrm{id}\otimes S)\circ\Delta.$$



Here $m : \hat{U}_t \hat{\otimes} \hat{U}_t \to \hat{U}_t$ is the *multiplication* and $u : \mathbb{C} \to \hat{U}_t$ is the *unit*. Namely, it holds that

$$\sum_i S(h_i^{(1)}) h_i^{(2)} = \varepsilon(h) \cdot 1 = \sum_i h_i^{(1)} S(h_i^{(2)}).$$

The operator S is an *anti-homomorphism*, and also *anti-cohomomorphism* that is,

$$S \circ m = m \circ (S \otimes S) \circ \sigma, \qquad S \circ u = u,$$
$$\Delta \circ S = \sigma \circ (S \otimes S) \circ \Delta, \qquad \varepsilon \circ S = \varepsilon,$$

where $\sigma(h_1 \otimes h_2) = h_2 \otimes h_1$ is the *flip*. Further, it holds that

$$S \circ * \circ S \circ * = \mathrm{id}.$$

Hence $\hat{U}_t$ becomes a $*$-*Hopf algebra*.

11.2 Tensor Product Representation

Take two U_q-modules V_1 and V_2. Here, for U_q-module V, we mean that V is an Hilbert space with $*$-representation $\mathrm{U}_q \to \mathrm{End}(V)$. Then the tensor product $V_1 \otimes V_2$ is naturally U_q-module via the comultiplication Δ;

$$\mathrm{U}_q \xrightarrow{\Delta} \mathrm{U}_q \otimes \mathrm{U}_q \curvearrowright V_1 \otimes V_2.$$

Hence we can take the tensor product in the category of U_q-modules. More explicitly, let v_i be basis of V_i $(i = 1, 2)$. Then the U_q-action on $V_1 \otimes V_2$ is given by

$$\begin{aligned} q^{\pm\frac{H}{2}}(v_1 \otimes v_2) &= q^{\pm\frac{H}{2}} v_1 \otimes q^{\pm\frac{H}{2}} v_2, \\ Y_\pm(v_1 \otimes v_2) &= v_1 \otimes Y_\pm v_2 + Y_\pm v_1 \otimes q^{\frac{H}{2}} v_2. \end{aligned}$$

Now take $V_1 = V^\alpha$ and $V_2 = V^\beta$ where V^α is the α-highest and V^β is the β-highest weight representation of U_q. Then $V^\alpha \otimes V^\beta$ is again a U_q-module and we have an orthogonal basis $\{v_n^\alpha \otimes v_m^\beta\}_{m,n\geq 0}$ of $V^\alpha \otimes V^\beta$ with the norm

$$||v_n^\alpha \otimes v_m^\beta||_{V^\alpha \otimes V^\beta} = ||v_n^\alpha||_{V^\alpha} ||v_m^\beta||_{V^\beta} = C_q^\alpha(n) C_q^\beta(m).$$

The U_q-action on these basis are explicitly given as follows;

$$\begin{cases} q^{\pm\frac{H}{2}}(v_n^\alpha \otimes v_m^\beta) = q^{\pm(\frac{\alpha+\beta}{2}+n+m)} \cdot v_n^\alpha \otimes v_m^\beta, \\ Y_+(v_n^\alpha \otimes v_m^\beta) = q^{\frac{\beta}{2}}[m+1]_q \cdot v_n^\alpha \otimes v_{m+1}^\beta + q^{\frac{\alpha+\beta}{2}+m}[n+1]_q \cdot v_{n+1}^\alpha \otimes v_m^\beta, \\ Y_-(v_n^\alpha \otimes v_m^\beta) = q^{-\frac{\beta}{2}}[\beta+m-1]_q \cdot v_n^\alpha \otimes v_{m-1}^\beta + q^{\frac{\beta-\alpha}{2}+m}[\alpha+n-1]_q \cdot v_{n-1}^\alpha \otimes v_m^\beta. \end{cases}$$

Let

$$v_{(m),0}^{(\alpha)\beta} := \sum_{i+j=m} c_{i,j} \cdot v_i^\alpha \otimes v_j^\beta.$$

This is a general vector of weight $\alpha + \beta + 2m$ because $q^{\frac{H}{2}} v_{(m),0}^{(\alpha)\beta} = q^{\frac{\alpha+\beta+2m}{2}} v_{(m),0}^{(\alpha)\beta}$. Consider the equation

$$Y_- v_{(m),0}^{(\alpha)\beta} = 0, \tag{11.4}$$

which says that $v_{(m),0}^{(\alpha)\beta}$ is the highest weight vector. It can be written as

$$\sum_{i+j=m} c_{i,j}\Big(q^{-\frac{\beta}{2}}[\beta+j-1]_q \cdot v_i^\alpha \otimes v_{i-1}^\beta + q^{\frac{\beta-\alpha}{2}+j}[\alpha+i-1]_q \cdot v_{i-1}^\alpha \otimes v_j^\beta\Big) = 0$$

and is also equivalent to the following equation for the coefficient $c_{i,j}$;

$$c_{i,j} = -c_{i-1,j+1} q^{-\beta-j+\frac{\alpha}{2}} \left(\frac{1-q^{\beta+j}}{1-q^{\alpha+i-1}} \right).$$

One can easily obtain the solution $c_{i,j}$ which is unique up to a constant. Therefore there exists a unique solution $v^{(\alpha)\beta}_{(m),0}$ of (11.4), that is, a $(\alpha+\beta+2m)$-highest weight vector. Actually, it is given by

$$v^{\alpha,\beta}_{(m),0} = \sum_{i+j=m} (-1)^j q^{\frac{j(j-1)}{2}+j\beta+i\frac{\alpha}{2}} \frac{\zeta_q(\alpha+m)}{\zeta_q(\alpha+i)} \frac{\zeta_q(\beta+m)}{\zeta_q(\beta+j)} \cdot v^\alpha_i \otimes v^\beta_j$$

As a corollary, we have the following irreducible decomposition

$$V^\alpha \otimes V^\beta \simeq \bigoplus_{m\geq 0} V^{\alpha+\beta+2m}.$$

Now in each space $V^{\alpha+\beta+2m}$ we have the $(\alpha+\beta+2m)$-highest weight vector $v^{\alpha,\beta}_{(m),0}$. We next get another basis. Let

$$v^{(\alpha)\beta}_{(m),n} := (-1)^n \frac{q^{-(\frac{\alpha+\beta}{2}+m)n}}{[n]_q!} Y^n_+ v^{(\alpha)\beta}_{(m),0}.$$

Then it is clear that $v^{(\alpha)\beta}_{(m),n} \in V^{\alpha+\beta+2(m+n)}$. Note that $Y^n_+ = \sum_{i+j=n} \begin{bmatrix} n \\ i \end{bmatrix}_q (1\otimes Y^j_+)(Y^i_+ \otimes q^{i\frac{H}{2}})$ by the q-binomial theorem. Hence we have

$$\begin{aligned}
v^{(\alpha)\beta}_{(m),n} &= (-1)^n q^{-(\frac{\alpha+\beta}{2}+m)n} \sum_{i+j=n} \frac{1}{[i]_q![j]_q!} (1\otimes Y^j_+)(Y^i_+\otimes q^{i\frac{H}{2}}) v^{\alpha,\beta}_{(m),0} \\
&= (-1)^n q^{-(\frac{\alpha+\beta}{2}+m)n} \sum_{i_1+j_1=n} \sum_{i_0+j_0=m} \frac{(-1)^{j_0} q^{\frac{j_0(j_0-1)}{2}+j_0\beta+i\frac{\alpha}{2}}}{[i_1]_q![j_1]_q!} \\
&\quad \times \frac{\zeta_q(\alpha+m)}{\zeta_q(\alpha+i_0)} \frac{\zeta_q(\beta+m)}{\zeta_q(\beta+j_0)} (1\otimes Y^{j_1}_+)(Y^{i_1}_+ \otimes q^{i_1\frac{H}{2}}) v^\alpha_{i_0} \otimes v^\beta_{j_0} \\
&= q^{-(\frac{\alpha+\beta}{2}+m)n} \sum_{i+j=m+n} q^{i\frac{\alpha}{2}+n\frac{\beta}{2}} \cdot v^\alpha_i \otimes v^\beta_j \\
&\quad \times \Bigg(\sum_{i_0+j_0=m} \begin{bmatrix} i \\ i_0 \end{bmatrix}_q \begin{bmatrix} j \\ j_0 \end{bmatrix}_q (-1)^{j_0} q^{\frac{j_0(j_0-1)}{2}+j_0\beta+j_0(i-i_0)} \\
&\qquad \times \frac{\zeta_q(\alpha+m)}{\zeta_q(\alpha+i_0)} \frac{\zeta_q(\beta+m)}{\zeta_q(\beta+j_0)} \Bigg)
\end{aligned}$$

since

$$\begin{aligned}(1\otimes Y_+^{j_1})(Y_+^{i_1}\otimes q^{i_1\frac{H}{2}})v_{i_0}^{\alpha}\otimes v_{j_0}^{\beta}\\ &= (-1)^n q^{i_1(\frac{\beta}{2}+j_0)}q^{i_1\frac{\alpha}{2}+j_1\frac{\beta}{2}}[i_0+1]_q\cdots[i_0+i_1]_q[j_0+1]_q\\ &\quad\cdots[j_0+j_1]_q\cdot v_{i_0+i_1}^{\alpha}\otimes v_{j_0+j_1}^{\beta}.\end{aligned}$$

Therefore we have

$$v_{(m),n}^{(\alpha)\beta}=\frac{\zeta_q(\alpha+\beta+n+2m)}{\zeta_q(\alpha+\beta+n+m)}q^{-mn+m\beta}\sum_{i+j=m+n}q^{\frac{\alpha}{2}(i-n)}\varphi_{q(m+n),m}^{(\alpha)\beta}(i,j)\cdot v_i^{\alpha}\otimes v_j^{\beta}.$$

Here $\varphi_{q(m+n),m}^{(\alpha)\beta}$ is the q-Hahn basis. Hence we obtain the new basis $\{v_{(m),n}^{(\alpha)\beta}\}_{m,n\geq 0}$ of the tensor product representation $V^{\alpha}\otimes V^{\beta}\simeq\bigoplus_{m\geq 0}V^{\alpha+\beta+2m}$. This shows that the Clebsch–Gordan coefficients, which is the matrix coefficients of the change of basis from $\{v_i^{\alpha}\otimes v_j^{\beta}\}_{i,j\geq 0}$ to $\{v_{(m),n}^{(\alpha)\beta}\}_{m,n\geq 0}$, are essentially given by the q-Hahn basis.

Note that, by simple calculations, the norm of the basis $v_{(m),n}^{(\alpha)\beta}$ is expressed as

$$\begin{aligned}||v_{(m),0}^{(\alpha)\beta}||^2_{V^{\alpha}\otimes V^{\beta}} &= \frac{\zeta_q(1)}{\zeta_q(1+m)}\frac{\zeta_q(\alpha+m)}{\zeta_q(\alpha)}\frac{\zeta_q(\beta+m)}{\zeta_q(\beta)}\frac{\zeta_q(\alpha+\beta+2m-1)}{\zeta_q(\alpha+\beta+m-1)},\\ ||v_{(m),n}^{(\alpha)\beta}||^2_{V^{\alpha}\otimes V^{\beta}} &= C_q^{\alpha+\beta+2m}(n)||v_{(m),0}^{\alpha,\beta}||^2_{V^{\alpha}\otimes V^{\beta}}.\end{aligned}$$

Considering the basis of the space $V^{\alpha+\beta+2N}$, we obtain the square matrix M of size $(N+1)$ which changes two orthonormal basis

$$\left\{\frac{v_i^{\alpha}\otimes v_j^{\beta}}{C_q^{\alpha}(i)^{\frac{1}{2}}C_q^{\beta}(j)^{\frac{1}{2}}}\right\}_{\substack{0\leq i,j\leq N\\ i+j=N}}\quad\text{and}\quad\left\{\frac{v_{(m),n}^{(\alpha)\beta}}{C_q^{\alpha+\beta+2m}(n)^{\frac{1}{2}}||v_{(m),0}^{(\alpha)\beta}||_{V^{\alpha}\otimes V^{\beta}}}\right\}_{\substack{0\leq m,n\leq N\\ m+n=N}}.$$

Actually, M is explicitly given as follows;

$$M=\left\{\frac{\varphi_{q(N),m}^{(\alpha)\beta}(i,j)}{||\varphi_{q(N),m}^{(\alpha)\beta}(i,j)||_{H_{q(N)}^{(\alpha)\beta}}}\tau_{q(N)}^{(\alpha)\beta}(i,j)^{\frac{1}{2}}\right\}_{\substack{0\leq i\leq N,\, i+j=N\\ 0\leq m\leq N}}.$$

Here i,j denote the geometrical coordinate and m the spectral parameter. This is an orthogonal matrix. Namely, we have $M\cdot M^t=I_{(N+1)\times(N+1)}=M^t\cdot M$. These two equality are translated as orthogonality and "dual orthogonality" relations for the $\varphi_{q(N),m}^{(\alpha)\beta}$:

$$\sum_{i+j=N}\varphi_{q(N),m_1}^{(\alpha)\beta}(i,j)\varphi_{q(N),m_2}^{(\alpha)\beta}(i,j)\tau_{q(N)}^{(\alpha)\beta}(i,j)=\delta_{m_1,m_2}||\varphi_{q(N),m_1}^{(\alpha)\beta}||^2_{H_{q(N)}^{(\alpha)\beta}},$$

$$\sum_{0\leq m\leq N}\frac{\varphi_{q(N),m}^{(\alpha)\beta}(i_1,j_1)\varphi_{q(N),m}^{(\alpha)\beta}(i_2,j_2)}{||\varphi_{q(N),m}^{(\alpha)\beta}||^2_{H_{q(N)}^{(\alpha)\beta}}}=\delta_{i_1,i_2}\delta_{j_1,j_2}\frac{1}{\tau_{q(N)}^{(\alpha)\beta}(i_1,j_1)}.$$

11.3 The Universal *R*-Matrix

The algebra U_q is not co-commutative in the sense that $\Delta \neq \sigma \circ \Delta$ where $\sigma(h_1 \otimes h_2) = h_2 \otimes h_1$ is the flip. However, in a certain sense, it is not far from being co-commutative. Namely, there exists a "universal R-matrix" R, which is an element of $\widehat{\mathrm{U}_q \otimes \mathrm{U}_q} = \hat{\mathrm{U}}_q \hat{\otimes} \hat{\mathrm{U}}_q$ and is explicitly given as

$$R = q^{H \otimes \frac{H}{4}} \sum_{n \geq 0} \frac{(1-q)^n}{[n]_q!} Y_+^n \otimes q^{-n\frac{H}{2}} Y_-^n.$$

One can show that R is invertible in $\hat{\mathrm{U}}_q \hat{\otimes} \hat{\mathrm{U}}_q$. In fact, we have

$$R^{-1} = \left(\sum_{n \geq 0} \frac{(1-q)^n}{[n]_q!} (-1)^n q^{\frac{n(n-1)}{2}} Y_+^n \otimes q^{-n\frac{H}{2}} Y_-^n \right) q^{-H \otimes \frac{H}{4}}.$$

The important point is that the inner conjugation by R repairs the lost co-commutativity. Namely, for any $h \in \mathrm{U}_q$, we have

$$\sigma \circ \Delta(h) = R \cdot \Delta(h) \cdot R^{-1}. \tag{11.5}$$

This can be shown by directly checking for the generators $h = q^{\pm\frac{H}{2}}$, Y_+ and Y_- of U_q.

Now take two U_q-modules V_1 and V_2. Then we have the isomorphism

$$R^{V_1,V_2} : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1, \qquad v_1 \otimes v_2 \longmapsto \sigma\big(R(v_1 \otimes v_2)\big).$$

Note that this is an isomorphism of U_q-modules. Indeed, from (11.5), we have for $v_1 \otimes v_2 \in V_1 \otimes V_2$ and $h \in \mathrm{U}_q$ that

$$\begin{aligned} R^{V_1,V_2}\Delta(h) v_1 \otimes v_2 &= \sigma\big(R(\Delta(h) v_1 \otimes v_2)\big) = \sigma\big(R\Delta(h)R^{-1}R(v_1 \otimes v_2)\big) \\ &= \sigma\big((\sigma\Delta(h))R(v_1 \otimes v_2)\big) = \Delta(h)\sigma R(v_1 \otimes v_2) \\ &= \Delta(h) R^{V_1,V_2} v_1 \otimes v_2. \end{aligned}$$

This shows that R^{V_1,V_2} commutes with the action of $\Delta(h)$. Further in $\hat{\mathrm{U}}_q \hat{\otimes} \hat{\mathrm{U}}_q \hat{\otimes} \hat{\mathrm{U}}_q$, we have

$$\begin{aligned} (\Delta \otimes \mathrm{id})R &= R_{13} \cdot R_{23}, \\ (\mathrm{id} \otimes \Delta)R &= R_{13} \cdot R_{12}, \end{aligned}$$

where R_{ij} is the image of R under the embedding of $\hat{\mathrm{U}}_q \hat{\otimes} \hat{\mathrm{U}}_q$ into the i-th and j-th component in $\hat{\mathrm{U}}_q \hat{\otimes} \hat{\mathrm{U}}_q \hat{\otimes} \hat{\mathrm{U}}_q$, and similarly

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{11.6}$$

The (11.6) is called the quantum Yang–Baxter equation. The meaning of the equation is as follows; Take three U_q-modules V_1, V_2 and V_3 (see Fig. 11.1). Then (11.6) says that as operators from $V_1 \otimes V_2 \otimes V_3$ to $V_3 \otimes V_2 \otimes V_1$ we have the braid relation

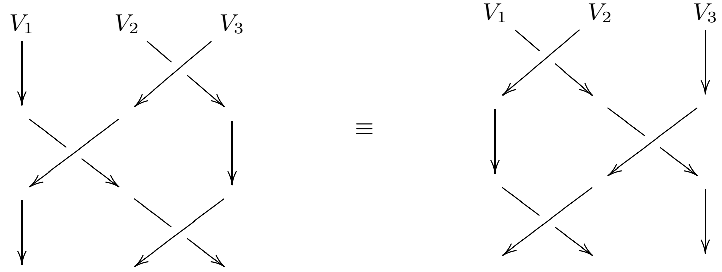


Fig. 11.1. The quantum Yang–Baxter equation

$$\begin{aligned}&(\mathrm{id}_{V_3}\otimes R^{V_1,V_2})(R^{V_1,V_3}\otimes \mathrm{id}_{V_2})(\mathrm{id}_{V_1}\otimes R^{V_2,V_3})\\&\quad=(R^{V_2,V_3}\otimes \mathrm{id}_{V_1})(\mathrm{id}_{V_2}\otimes R^{V_1,V_3})(R^{V_1,V_2}\otimes \mathrm{id}_{V_3}).\end{aligned}$$

Further R satisfies more equations;

$$\begin{aligned}&S\otimes \mathrm{id}(R)=R^{-1},\quad \mathrm{id}\otimes S(R^{-1})=R\quad \text{hence}\quad S\otimes S(R)=R,\\&(*\otimes *)R=\tau(R)\quad \text{hence}\quad (R^{V_1,V_2})^*=R^{V_2,V_1}.\end{aligned}$$

Now take two highest weight representations $V_1=V^\alpha$ and $V_2=V^\beta$. As we saw in the last subsection, the tensor product representation $V^\alpha\otimes V^\beta$ has two basis, that is, $\{v^{(\alpha)\beta}_{(m),n}\}_{m,n\geq 0}$ and $\{v^\alpha_i\otimes v^\beta_j\}_{i,j\geq 0}$. Here let us consider the operator

$$R^{\alpha,\beta}:=R^{V^\alpha,V^\beta}:V^\alpha\otimes V^\beta\longrightarrow V^\beta\otimes V^\alpha.$$

Using the explicit formula for $R^{\alpha,\beta}$, we can calculate the action of $R^{\alpha,\beta}$ on both the basis $v^{(\alpha)\beta}_{(m),n}$ and the basis $v^\alpha_i\otimes v^\beta_j$ as follows;

$$\begin{aligned}R^{\alpha,\beta}v^{(\alpha)\beta}_{(m),n}&=(-1)^m q^{\frac{m(m-1)}{2}+\frac{m(\alpha+\beta)}{2}+\frac{\alpha\beta}{4}}v^{(\beta)\alpha}_{(m),n},\\R^{\alpha,\beta}(v^\alpha_i\otimes v^\beta_j)&=q^{(\frac{\alpha}{2}+i)(\frac{\beta}{2}+j)}\sum_{0\leq k\leq j}\begin{bmatrix}i+k\\i\end{bmatrix}_q q^{-k(\frac{\beta}{2}+i)}\frac{\zeta_q(\beta+j)}{\zeta_q(\beta+j-k)}v^\beta_{j-k}\otimes v^\alpha_{i+k}.\end{aligned}$$

Comparing these matrices of $R^{\alpha,\beta}$ using the interchange of basis matrix M, which is essentially given by the q-Hahn basis, we have the following identity;

$$\begin{aligned}&(-1)^m q^{\frac{m(m-1)}{2}+m\alpha-i_0j_0}\varphi^{(\beta)\alpha}_{q(N),m}(i_0,j_0)\\&\quad=\sum_{i_1+j_1=N}\begin{bmatrix}j_0\\j_0-i_1\end{bmatrix}_q q^{i_1\beta}\frac{\zeta_q(\beta+j_1)}{\zeta_q(\beta+i_0)}\varphi^{(\alpha)\beta}_{q(N),m}(i_1,j_1).\end{aligned}$$

This means that the universal R-matrix $R^{\alpha,\beta}$ repairs the lost symmetry of the non-symmetric q-β-chain. Remark that our chains are not symmetric in the parameter α and β.

A

Problems and Questions

Chapter 3

1. (Section 3.3) The symmetric q-β-chain is defined by the following Figure A.1.

   Its real limit is the η-β-chain (which is symmetric in (α, β)), and its p-adic limit is the symmetric p-β-chain (Sect. 2.1.3). Understand the Martin kernel and the boundary of this symmetric q-β-chain. Note that for this chain $(P^*)\delta_{(0,0)}$ is a probability measure supported at $\{(i,j) \mid N \le i+j, \max\{i,j\} \le N\}$, its real limit is supported at $\{(i,j) \mid i+j = N\}$, and its p-adic limit supported at $\{(i,j) \mid \max\{i,j\} = N\}$.

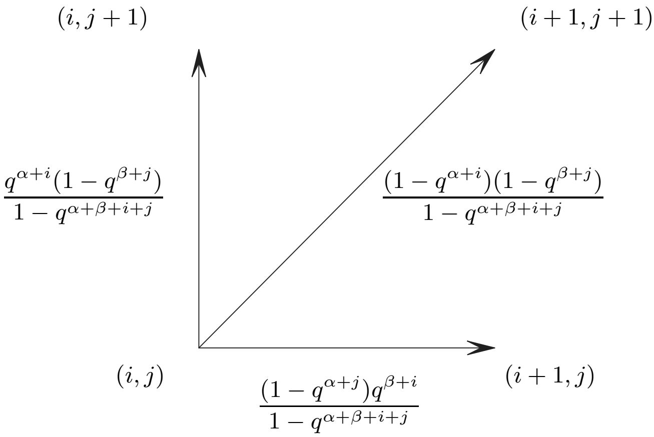


Fig. A.1. The symmetric q-β-chain

Chapter 4

1. (Section 4.5) Find the real γ-chain. Find the η-finite Laguerre basis.

Chapter 6

1. (Section 6.1) Find the q-pure basis. See remark on pp. 112–113 and p. 128 of [Har5].

Chapter 8

1. Determine the idempotent explicitly.

Chapter 9

1. Determine the idempotent explicitly (i.e., find the inverse matrix $(A^*_{\lambda,\alpha})$ of $(A_{\lambda,\alpha})$).

Chapter 10

1. Get the direct proof of Theorem 10.3.1 without going through the Koornwinder polynomials.
2. There is no q-chain and no η-chain which is analogue of the p-adic chain (like we had in the rank 1 case). The problem is that dividing $GL_d/B_m = \text{Grass}(m,d)$ by B_m kills the Schubert cells but $B_{1,\dots,1}\backslash GL_d/B_m$ preserves them. However $B_{1,\dots,1}\backslash GL_d/B_m\big(\mathbb{Z}/p^N\big)$ depends on p (and not just on N). Does there exist some combinatorial quotient $\mathcal{C}_N^{d,m}$ of $B_{1,\dots,1}\backslash GL_d/B_m\big(\mathbb{Z}/p^N\big)$ and Markov chain on $\sqcup_N \mathcal{C}_N^{d,m}$? Namely, begin at the closed point and look in which Schubert cell you fall modulo p^N, $N = 1, 2, \dots$ (eventually end up in big open cell with probability 1).
3. There is no explicit description of the idempotent such that

$$\varphi_\lambda = (D^+)^{\lambda_1-\lambda_2}\rho_{m-1}(D^+)^{\lambda_2-\lambda_3}\rho_{m-2}\cdots\rho_2(D^+)^{\lambda_m}\mathbf{1}$$

as we had for intertwiner $GL_d/B_{1,d-1} \longrightarrow GL_d/B_{1,\dots,1}$ in Chapter 7. We need a factorization of the (radial parts) Laplacian Δ as $\Delta = D^+D$. More generally, let Δ be the Koornwinder 2-nd order differential operator,

$$\Delta = \sum_{1\le j\le m} \phi_j^+(T_j^+ - I) + \phi_j^-(T_j^- - I),$$

where $T_j^{\pm}\varphi(y_1,\ldots,y_m) := \varphi(y_1,\ldots,q^{\pm 1}y_j,\ldots,y_m)$ and

$$\phi_j^+ := \frac{\prod_{0\le i\le 3}(1-a_i y_j)}{(1-y_j^2)(1-qy_j^2)}\prod_{i\neq j}\frac{(1-ty_iy_j)(1-ty_i^{-1}y_j)}{(1-y_iy_j)(1-y_i^{-1}y_j)},$$

$$\phi_j^- := \frac{\prod_{0\le i\le 3}(a_i - y_j)}{(1-y_j^2)(q-y_j^2)}\prod_{i\neq j}\frac{(t-y_iy_j)(t-y_i^{-1}y_j)}{(1-y_iy_j)(1-y_i^{-1}y_j)}$$

with $a_i = q^{\alpha_i}$ $(i = 0,\ldots,3)$ and $t = q^{\gamma}$ being parameters. In the case of $m = 1$, Δ is the Askey–Wilson operator which depends on parameters a_i ($i = 0,\ldots,3$ but not on $t = q^{\gamma}$) and there is such a factorization $\Delta = D^+D$ (cf. [Har5] pp. 126–127). The Askey–Wilson polynomial φ_λ can be written as $\varphi_\lambda = (D^+)^\lambda\mathbf{1}$ for $\lambda \in \mathbb{N}$.

4. Normalize the q-algebra of this chapter using the non-symmetric q-numbers (as we do in Chapter 11) so as to have a p-adic limit of the algebra.

Chapter 11

1. Is there a more refined p-adic limit, one that will preserve more of the SL_2-structure than the limit of (11.2)?

B

Orthogonal Polynomials

Let p be a prime number. Given a probability measure μ on $\mathbb{Z}_p$, we have its image on $\mathbb{Z}/p^N$ for any N, $\mu_N(a) = \mu(a + p^N\mathbb{Z}_p)$, and the isometric embeddings, and dual orthogonal projection:

$$H_N = \ell^2(\mathbb{Z}/p^N, \mu_N) \underset{\hookrightarrow}{\twoheadleftarrow} L^2(\mathbb{Z}_p, \mu) = H$$

We want the real analogue of this, and the idea is simple: We replace the locally constant function (defined modulo p^N) by the polynomials (of degree $\leq N$). We shall show that the theory of orthogonal polynomials gives the real analogue.

Let μ be a probability measure on $[-1, 1]$, $H = L^2\big([-1, 1], \mu\big)$. Applying the Gram–Schmidt process to the monomials $1, x, x^2, \ldots$, we get a sequence of polynomials which are orthogonal in H:

$$\begin{aligned}
p_0(x) &= 1,\\
p_1(x) &= x - (x, 1) \cdot 1,\\
p_2(x) &= x^2 - \frac{(x^2, p_1)}{(p_1, p_1)} p_1 - \frac{(x^2, 1)}{(1, 1)} \cdot 1,\\
&\vdots\\
p_n(x) &= x^n - \frac{(x^n, p_{n-1})}{(p_{n-1}, p_{n-1})} p_{n-1} - \cdots - \frac{(x^2, 1)}{(1, 1)} \cdot 1.
\end{aligned}$$

Here $(\cdot, \cdot)$ denotes the inner product of H; $(\varphi_1, \varphi_2) := \int_{-1}^{1} \varphi_1(x)\overline{\varphi_2(x)}\mu(dx)$. We here normalize the orthogonal polynomials $p_n(x)$ to have the leading coefficient 1. We have

Theorem B.1.

$$p_n(x) = \det \begin{pmatrix} (1,1) & (1,x) & \cdots & (1,x^n) \\ (x,1) & (x,x) & \cdots & (x,x^n) \\ \vdots & \vdots & & \vdots \\ (x^{n-1},1) & (x^{n-1},x) & \cdots & (x^{n-1},x^n) \\ 1 & x & \cdots & x^n \end{pmatrix} \cdot \frac{1}{G_{n-1}},$$

where $G_{n-1} = \det\big((x^i, x^j)\big)_{0 \le i,j \le n-1}$.

Proof. Indeed, the inner product of the determinant with x^j is the same determinant with the bottom row replaced by (x^i, x^j), and for $j < n$ this row already appears in the determinant, whence it vanishes. So the determinant above is a polynomial of degree n which is orthogonal to all polynomials of degree $< n$ and it has the leading coefficient 1 because of the $1/G_{n-1}$ normalization factor. □

Note that if we denote the moments of the measure μ by

$$c_n := (x^n, 1) = \int_{-1}^{1} x^n \mu(dx),$$

we have $(x^i, x^j) = c_{i+j}$.

Theorem B.2. *All the zeros of* $p_n(x)$ *are simple and are contained in* $(-1,1)$.

Proof. Otherwise $p_n(x)$ changes sign in $(-1,1)$ only in $m<n$ point $\alpha_1, \ldots, \alpha_m$. Then $\pm p_n(x) \cdot \prod_{j=1}^m (x - \alpha_j) \ge 0$ in $(-1,1)$, whence $\pm\big(p_n, \prod_{j=1}^m (x-\alpha_j)\big) > 0$. This contradicts the orthogonality of $p_n(x)$ to polynomials of degree $< n$. □

Theorem B.3. *We have the recursion equation*

$$p_{n+1}(x) = (x + b_n)p_n(x) - d_n p_{n-1}(x),$$

where

$$d_n = \frac{h_n}{h_{n-1}}, \qquad h_n = ||p_n||^2 = (p_n, p_n),$$
$$b_n = k_{n+1} - k_n, \qquad p_n(x) \equiv x^n + k_n x^{n-1} \pmod{x^{n-2}}.$$

Proof. The polynomial $p_{n+1} - x \cdot p_n$ is of degree $\le n$ and is orthogonal to polynomials of degree $\le n-2$, hence has the form $b_n \cdot p_n - d_n \cdot p_{n-1}$. We get

$$d_n = \frac{(x \cdot p_n - p_{n+1}, p_{n-1})}{(p_{n-1}, p_{n-1})} = \frac{(p_n, x \cdot p_{n-1})}{(p_{n-1}, p_{n-1})} = \frac{(p_n, p_n + (\text{degree} < n))}{(p_{n-1}, p_{n-1})} = \frac{(p_n, p_n)}{(p_{n-1}, p_{n-1})},$$

and modulo x^{n-1}

$$b_n \cdot x^n \equiv b_n \cdot p_n(x) \equiv p_{n+1}(x) - x \cdot p_n(x)$$
$$\equiv (x^{n+1} + k_{n+1}x^n) - x(x^n + k_n x^{n-1}) \equiv (k_{n+1} - k_n) \cdot x^n.$$

□

Corollary B.4 (Christoffel–Derboux).

$$D_n(x,y) := \sum_{j=0}^{n} \frac{1}{h_j} p_j(x)p_j(y) = \frac{1}{h_n} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}, \quad (x \neq y).$$

and for $y \to x$:

$$D_n(x,x) := \sum_{j=0}^{n} \frac{1}{h_j} p_j(x)^2 = \frac{1}{h_n}\big(p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\big).$$

Proof. By induction on n. The induction step follows from the equality

$$\begin{aligned}
&h_n(x-y)\big(D_n(x,y) - D_{n-1}(x,y)\big) = (x-y)p_n(x)p_n(y) \\
&\quad = \Big(p_{n+1}(x) - b_n p_n(x) + \frac{h_n}{h_{n-1}} p_{n-1}(x)\Big) p_n(y) \\
&\qquad - p_n(x)\Big(p_{n+1}(y) - b_n p_n(y) + \frac{h_n}{h_{n-1}} p_{n-1}(y)\Big) \\
&\quad = \big(p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)\big) - \frac{h_n}{h_{n-1}}\big(p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)\big).
\end{aligned}$$

□

Remark B.5. For $\varphi \in H$, its orthogonal projection $D_n\varphi$ to the subspace $H_n := \mathrm{Span}\{1, x, \ldots, x^n\}$ of H spanned by polynomials of degree $\leq n$ is given by

$$\begin{aligned}
D_n\varphi(y) &:= \sum_{j=0}^{n} \frac{(\varphi, p_j)}{(p_j, p_j)} p_j(y) \\
&= \int_{-1}^{1} \varphi(x) \sum_{j=0}^{n} \frac{p_j(x)p_j(y)}{h_j} \mu(dx) \\
&= \int_{-1}^{1} \varphi(x) D_n(x,y)\mu(dx) = (\varphi, D_n(\cdot, y)).
\end{aligned}$$

Corollary B.6. *$p_{n+1}(t)$ is the characteristic polynomial of the operator $D_n x$ of multiplication by x restricted to H_n;*

$$p_{n+1}(t) = \det\big(t \cdot I_n - D_n x \,\big|\, H_n\big).$$

Proof. The matrix corresponding to the operator $D_n x$ in the basis $\{p_0, p_1, \ldots, p_n\}$ of H_n is by the recursion $x \cdot p_j = p_{j+1} - b_j p_j + d_j p_{j-1}$, given by

$$\begin{pmatrix} -b_0 & d_1 & & & & \\ 1 & -b_1 & & \vdots & & \\ 0 & 1 & \ddots & d_j & \vdots & \\ \vdots & 0 & & -b_j & 0 & \vdots \\ & \vdots & & 1 & \ddots\ d_{n-1} & 0 \\ & & & 0 & -b_{n-1} & d_n \\ & & & \vdots & 1 & -b_n \end{pmatrix}.$$

Expanding the determinant of the characteristic polynomial by the last raw, we have

$$\begin{aligned} Q_{n+1}(t) :&= \det \begin{pmatrix} t+b_0 & -d_1 & & & & \\ -1 & t+b_1 & & \vdots & & \\ 0 & -1 & \ddots & -d_j & \vdots & \\ \vdots & 0 & & t+b_j & 0 & \vdots \\ & \vdots & & -1 & \ddots\ -d_{n-1} & 0 \\ & & & 0 & t+b_{n-1} & -d_n \\ & & & \vdots & -1 & t+b_n \end{pmatrix} \\ &= (t+b_n)Q_n(t) - d_n Q_{n-1}(t). \end{aligned}$$

Hence $Q_{n+1}(t) = p_{n+1}(t)$. □

Given a sequence of points $-1 < \alpha_0 < \alpha_1 < \cdots < \alpha_n < 1$, let

$$L(x) := \prod_{j=0}^{n}(x-\alpha_j) \quad \text{and} \quad L_j(x) := \frac{L(x)}{(x-\alpha_j)L'(\alpha_j)} = \prod_{i \neq j} \frac{(x-\alpha_i)}{(\alpha_j - \alpha_i)},$$

so that $L_j(\alpha_i) = \delta_{ij}$. Given a function $\varphi(x)$ on $[-1,1]$, we approximate it by the polynomial of degree $\leq n$,

$$\mathcal{L}_n \varphi(x) = \sum_{j=0}^{n} \varphi(\alpha_j) L_j(x).$$

Note that if $\varphi(x)$ is a polynomial of degree $\leq n$, then $\mathcal{L}_n\varphi(x) = \varphi(x)$ and hence

$$\int_{-1}^{1} \varphi(x)\mu(dx) = \int_{-1}^{1} \mathcal{L}_n\varphi(x)\mu(dx) = \sum_{j=0}^{n} \varphi(\alpha_j) \int_{-1}^{1} L_j(x)\mu(dx).$$

We do better if we choose the α_j's to be the zeros of $p_{n+1}(x)$, that is, $L(x) = p_{n+1}(x)$:

Theorem B.7 (Mechanical Quadrature). *For $L(x) = p_{n+1}(x)$, we have*

$$\int_{-1}^{1} \varphi(x)\mu(dx) = \int_{-1}^{1} \mathcal{L}_n\varphi(x)\mu(dx)$$

for all polynomials $\varphi(x)$ of degree $\leq 2n+1$.

Proof. $\varphi(x) - \mathcal{L}_n\varphi(x)$ is a polynomial of degree $\leq 2n+1$ and it vanishes at the α_j's, so

$$\varphi(x) - \mathcal{L}_n\varphi(x) = p_{n+1}(x)f(x)$$

where $f(x)$ is a polynomial of degree $\leq n$. Therefore

$$(\varphi - \mathcal{L}_n\varphi, 1) = (p_{n+1}f, 1) = (p_{n+1}, f) = 0. \qquad \square$$

Thus for $\{\alpha_j^{(n+1)}\}_{j=0,1,\ldots,n}$ the zeros of $p_{n+1}(x)$ and the Christoffel numbers

$$\lambda_j^{(n+1)} := \int_{-1}^{1} \frac{p_{n+1}(x)}{(x-\alpha_j^{(n+1)})p'_{n+1}(x)}\mu(dx) = \int_{-1}^{1} L_j(x)\mu(dx),$$

we have for any polynomial $\varphi(x)$ of degree $\leq 2n+1$

$$\int_{-1}^{1} \varphi(x)\mu(dx) = \int_{-1}^{1} \varphi(x)\mu_n(dx)$$

with the finite probability measure

$$\mu_n := \sum_{j=0}^{n} \lambda_j^{(n+1)} \delta_{\alpha_j^{(n+1)}}.$$

Remark that for $\varphi(x) = L_j(x)^2$ we get

$$\lambda_j^{(n+1)} = \int_{-1}^{1} L_j(x)^2 \mu_n(x) = \int_{-1}^{1} L_j(x)^2 \mu(dx) > 0,$$

and for $\varphi(x) = 1$,

$$\sum_{j=0}^{n} \lambda_j^{(n+1)} = \int_{-1}^{1} 1 \cdot \mu_n(x) = \int_{-1}^{1} 1 \cdot \mu(dx) = 1.$$

In particular, for polynomials φ_1, φ_2 of degree $\leq n$, we have

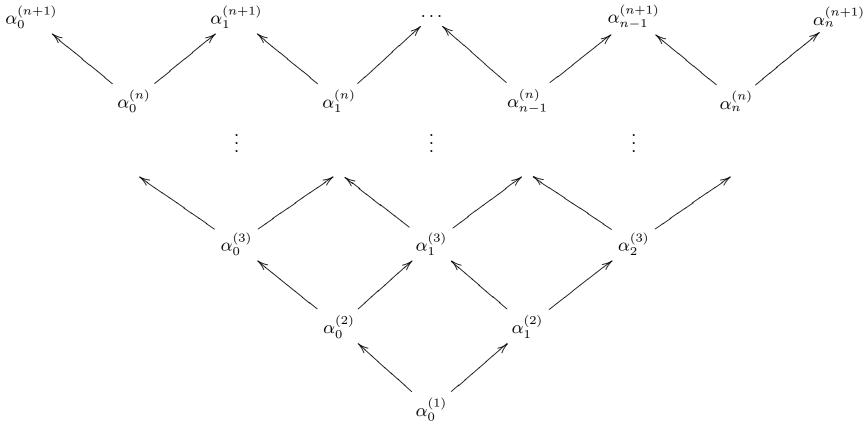
$$\int_{-1}^{1} \varphi_1(x)\overline{\varphi_2(x)}\mu_n(x) = \sum_{j=0}^{n} \lambda_j^{(n+1)} \varphi_1(\alpha_j^{(n+1)})\overline{\varphi_2}(\alpha_j^{(n+1)}) = (\varphi_1, \varphi_2).$$

Thus we can identify H_n isometrically with $\ell_2(\mu_n)$.

Remark also that the zeros $\alpha_j^{(n+1)}$ of $p_{n+1}(x)$ and the zeros $\alpha_j^{(n)}$ of $p_n(x)$, interlace:

$$-1 < \alpha_0^{(n+1)} < \alpha_0^{(n)} < \alpha_1^{(n+1)} < \alpha_1^{(n)} < \cdots < \alpha_{n-1}^{(n)} < \alpha_n^{(n+1)} < 1.$$

Thus we have the picture



This is the picture of our Markov chain.

The *classical* orthogonal polynomials $\{p_n(x)\}$ are characterized (modulo translation and dilation $\{p_n(x)\} \sim \{p_n(ax+b)\}$ changing the interval $[-1,1]$ to an arbitrary interval $[b-a, b+a]$) by any of the following equivalent condition:

1. Hahn: The sequence $\left\{p_n'(x) = \frac{\partial}{\partial x} p_n(x)\right\}_{n \geq 1}$ are again a sequence of orthogonal polynomials (with respect to another measure μ').
2. Bochner: $p_n(x)$ are the eigenfunctions of a second order differential operator

$$\Big(a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} + c(x)\Big)p_n(x) = \lambda_n p_n(x).$$

3. Tricomi: $p_n(x)$ can be expressed by a Rodriguez equation

$$p_n(x) = \frac{1}{\gamma_n}\frac{1}{\mu(x)}\frac{\partial^n}{\partial x^n}\mu(x) f(x)^n.$$

These polynomials are either the Jacobi polynomials $p_n^{\alpha,\beta}(x)$, $\mu^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$ $(\alpha, \beta > -1)$ or their $\beta \to \infty$ limit, the Laguerre polynomials, or their $\alpha = \beta \to \infty$ limit, the Hermite polynomials. Replacing $\frac{\partial}{\partial x}$ in the above by

$$D_q\varphi(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x},$$

we get a similar characterization of the q-classical orthogonal polynomials.

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